1. Current distribution in superconducting domains subject to transport current and external magnetic field - The equivalent electric network

Introduction

In this chapter we describe the model of the equivalent electric network, that is an integral method to solve numerically a general magnetoquasistatic problem.

In section 1.1 the mathematical description, based on the A-φ formulation, is given. Section 1.2 is dedicated to the description of the discretization technique. In order to stress the circuit interpretation of field problem, a graph is associated to the 3D mesh since the beginning. All the numerical details are worked out. In section 1.3 the reduction of the size of the solving system is dealt with, both by taking in account the eventual lower dimensionality of the problem and eliminating all the non essential variables by means of algebraic procedure. The circuit analogy is again stressed. A simple numerical example aimed to show how the complete model works is developed in section 1.4. Numerical results relative to some practical cases where an analytical solution is available are presented in section 1.5. In section 1.6 the generalization of the model to the case of composite superconducting materials, by means of a suitable material characteristic, is carried out. Finally, the possibility to relate the field (local) to the circuit (integral) quantities by means of higher order functions is discussed in section 1.7.

1.1 The mathematical formulation

Let us consider a system made of a superconducting domain (SC) with assigned time varying transport current and subject to the magnetic field produced by currents flowing outside. Let us limit our analysis to the cases of transport current and external magnetic field varying on characteristic time scales which are low enough compared with the time required by an electromagnetic wave to propagate over the entire extension of the domain. In general, if a systems is supplied only by slow varying sources, the energy is mainly stored either in magnetic or electric form rather than redistributes between the two equally; such a system is said to be *quasistatic*. A system with extension on the order of meters can be considered quasistatic if the sources change with time with frequencies up to tens of MHz. More in particular, since the considered system is made of superconductors, currents are allowed to flow very easily without the need of strong electric fields, therefore the magnetic field is mainly produced by electric currents, with negligible contribution of displacement currents and the predominant form of stored energy is magnetic in nature. When this condition holds the system is referred to as *magnetoquasistatic* [32,33].

In a magnetoquasistatic system made of magnetically homogeneous materials having magnetic permeability μ_0, the magnetic flux density **B** and the current density **J** are related through the Ampere law:

$$\nabla \times \mathbf{B}(\mathbf{x},t) = \mu_0 \mathbf{J}(\mathbf{x},t) \tag{1.1.1}$$

Vector **J** of equation (1.1.1) represents the *total* current density at any point, i.e. if point **x** lies inside the superconducting domain, vector **J** is the sum of the applied transport current and the shielding current induced in the superconductor that exhibit the Meissner effect [33,34]. From equation (1.1.1) it follows that, under the magnetoquasistatic approximation, the current density is a vector having zero divergence everywhere, i.e.

$$\nabla \cdot \mathbf{J}(\mathbf{x},t) = 0 \tag{1.1.2}$$

The magnetic flux density is always a zero divergence vector at any point, that is

$$\nabla \cdot \mathbf{B}(\mathbf{x}, t) = 0 \tag{1.1.3}$$

From equation (1.1.3) it follows that vector **B** can always be expressed as the curl of a regular[1] vector function **A** named vector magnetic potential [35]

$$\mathbf{B}(\mathbf{x}, t) = \nabla \times \mathbf{A}(\mathbf{x}, t) \tag{1.1.4}$$

Any other vector function **A’** obtained from **A** by just adding the gradient of an arbitrary regular scalar function still leads to the magnetic flux density **B** through equation (1.1.4). Among all this class of equivalent vector potentials the *suitable* one is selected by arbitrarily assigning its divergence.

By substituting equation (1.1.4) in eq. (1.1.1) and choosing the divergence of **A** to be equal to zero, the following vector Poisson equation is obtained

$$\nabla^2 \mathbf{A}(\mathbf{x}, t) = \mu_0 \mathbf{J}(\mathbf{x}, t) \tag{1.1.5}$$

By solving equation (1.1.5) the following expression of magnetic vector potential is obtained

$$\mathbf{A}(\mathbf{x}, t) = \frac{\mu_0}{4\pi} \int_{V_\infty} \frac{\mathbf{J}(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \tag{1.1.6}$$

where the integral is calculated over all the three dimensional space V_∞.

The local dependence of electric field **E** on the magnetic flux density **B** is expressed through the Faraday law:

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[1] A vector function **f**(r) is regular if it continuous in all the space and goes to zero as $1/r$ when r goes to infinite

$$\nabla \times \mathbf{E}(\mathbf{x},t) = -\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t} \tag{1.1.7}$$

By substituting equation (1.1.4) in the equation (1.1.7) and exchanging the curl and time derivative operators, it follows

$$\nabla \times \left(\mathbf{E}(\mathbf{x},t) + \frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t} \right) = \mathbf{0} \tag{1.1.8}$$

Thus vector field $\mathbf{E} + \partial\mathbf{A}/\partial t$ has zero curl everywhere and can be expressed as the gradient of a regular scalar function φ named scalar electric potential [35], therefore the electric field **E** is expressed as:

$$\mathbf{E}(\mathbf{x},t) = -\nabla\varphi(\mathbf{x},t) - \frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t} \tag{1.1.9}$$

Equation (1.1.9) is usually referred to as the $\mathbf{A} - \varphi$ formulation of magnetoquasistatics. By substituting in it expression (1.1.6) of the vector potential the following equation, relating electric field to current, is obtained

$$\mathbf{E}(\mathbf{x},t) = -\nabla\varphi(\mathbf{x},t) - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\int_{V_\infty} \frac{\mathbf{J}(\mathbf{x}',t)}{\left|\mathbf{x}-\mathbf{x}'\right|} d^3\mathbf{x}' \tag{1.1.10}$$

The electric field **E** at any point of the SC region is related to the local current density by means of the constitutive relation of the superconducting material. Equations (1.1.2) and (1.1.10), together with the constitutive relation of the superconducting material, form the basis of the model of the equivalent electric network which is developed in sections 1.2 and 1.3.

1.2 The discretized problem and the equivalent electric network

Let us consider a system made of a superconducting region (SC) with assigned time varying transport current and a current-driven normal conducting region (NC) placed in the empty space. The dimensions of the NC and SC domains and the frequency of variation of the impressed currents are such that the magnetoquasistatic approximation holds. Our goal is to determine the current density and the AC losses everywhere inside the superconducting domain at any instant.

Let us divide the superconducting region in a finite number N_E of three-dimensional elements. Let N_F be the number of faces of the discretization. Let us also define a normal unit vector for all faces. Depending on their position the faces can be distinguished as faces lying inside the SC domain or lying on the interface surfaces with the two poles of the generator which inject the transport current $I_{\mathrm{tr}}(t)$ in the SC domain. Let the number of these faces be equal to N_C. Furthermore, there is a number $(N_F - N_C)$ of faces lying on the boundary of the domain which, due to the fact that in magnetoquasistatics it is not possible to have non zero normal component of current density at the interface with non conducting media, have no current flowing through them. Let us assume all the N_C currents flowing through the faces of the discretized SC region and the $(N_E + 1)$ electric scalar potentials of the centers of the elements and of the positive pole of the generator as unknowns of the problem. All the physical quantities involved in the calculation have to be expressed as a function of them. We assume all the currents to be oriented according to the normal unit vector of the face, i. e. a positive current is given by a current density whose flux through the face, respect to the direction of the normal unit vector, is positive. In a first moment all the currents are assumed to be independent quantities which are allowed to take any value; the mathematical constrains for their physical consistency will be stated later. Let us indicate with $\mathbf{I}(t)$ and $\mathbf{V}(t)$ the vector of the N_C unknowns currents at time t.

In order to better fix the concepts it is convenient to refer to an example. Let us consider the cylindrical superconducting body, connected to a current generator and subject to the magnetic field produced by a current driven normal conducting coil, represented in figure 1.2.1. We assume a Cartesian coordinate system having the z axis parallel to axis of the cylinder and the origin coincident with its centre as reference system.

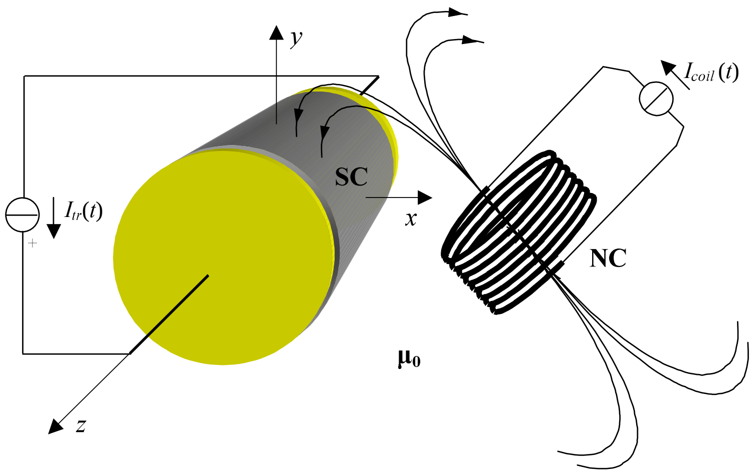

figure 1.2.1: superconducting cylinder with assigned transport current and subject to external magnetic field

The superconducting cylinder is connected to the current generator by means of two equipotential electrodes which span all the front and back cross section. We subdivide the cylindrical volume in a number of prisms with triangular basis. Since at this stage we are focusing only on the explanation of the numerical method we can refer to a coarse mesh, made of few elements, which however allow us to fully work out the numerical details without being too cumbersome to deal with. Moreover, it is worth to notice that there is no particular reason to use prisms with triangular basis for building the discretization instead of tetrahedrons, parallelepipeds, prisms with different basis or others. The only reason why they have been chosen is that such a prismatic mesh is easily obtained by "extruding" a two-dimensional mesh made of triangles. Let us adopt a three dimensional mesh made of 72 prisms, arranged in 3 groups of 24 prisms each. Any group of prisms, which cover a section spanning one third of the SC cylinder, is obtained by sliding the two dimensional triangular mesh of the cross section, made of 24 triangles, along the axis of the cylinder, as shown in figure 1.2.2.

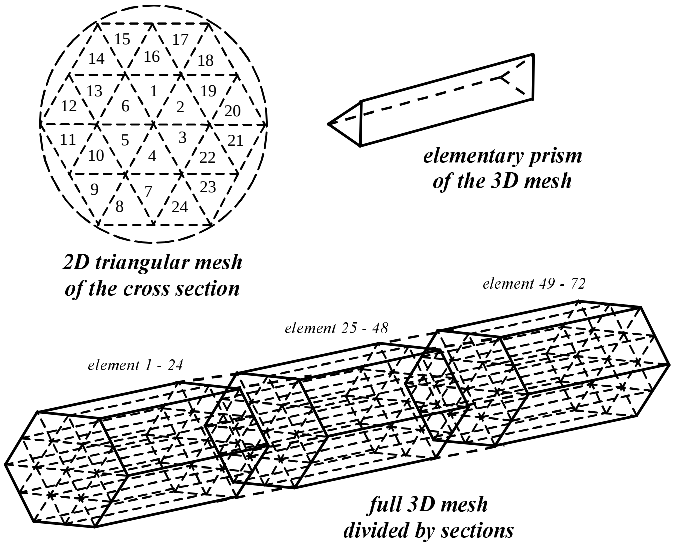

figure 1.2.2: mesh of the SC cylinder

Since the 2D mesh of the circular cross section is composed of 19 nodes and 42 segments, the full 3D mesh of the cylinder has 74 nodes, 126 (42 × 3) rectangular faces parallel to the sliding direction (z axis) and 96 (24 × 4) triangular faces orthogonal to it; the total number N_F of faces is equal to 222. Moreover, 12 of the 42 segments of the 2D mesh are edge segments, therefore the 3D mesh of the SC cylinder contains 36 (12 × 3) faces lying on the border with the empty space. It follows that the number N_C of faces of the mesh which can have a non zero current flowing through them is equal to 186, consequently the discretized problem expects 186 unknowns.

Let us now associate an oriented graph $\mathcal{G}$ to the mesh of the superconducting domain in the following way

- any of the centers of the N_E elements of the SC mesh corresponds to a node of the graph. Two additional nodes are provided for representing the two electrodes of the generator; the total number of nodes is equal to $N_E + 2$
- any of the N_C faces of the SC mesh with a current flowing through corresponds to a branch of the graph; the total number of branches is equal to N_C

- any branch is oriented according to normal unit vector of the corresponding face

Now on we agree to label the nodes corresponding to the positive and the negative electrode of the generator as the second last ((N_E+1)-*th*) and the last ((N_E+2)-*th*) node respectively. Even if not strictly necessary, this choice will result convenient later. The graph relative to the mesh of figure 1.2.2, containing 74 nodes and 186 branches, is represented in figure 1.2.3. The above labeling conventions are used.

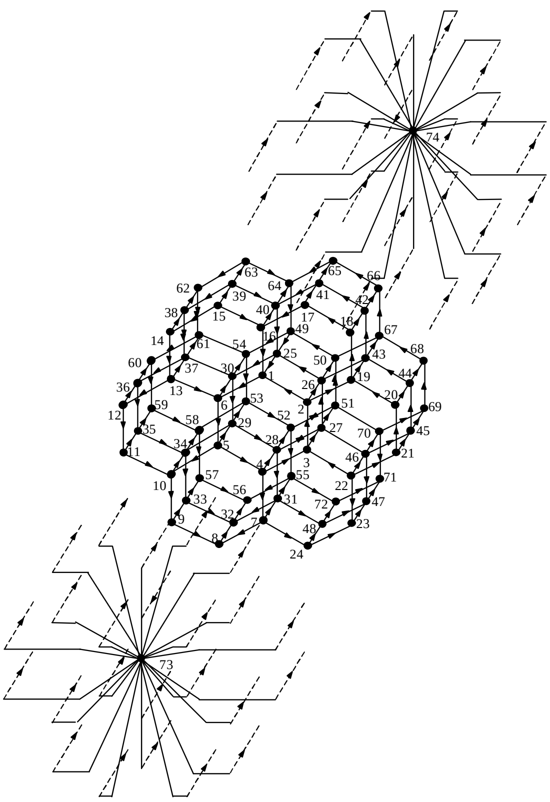

figure 1.2.3:graph associated to the 3D mesh of the SC cylinder

The nodes relative to the poles of the generator (73 and 74) have been sketched apart only for the sake of picture clarity. It is worth to notice that the complete graph is composed by the assembling of three sub-graphs with hexagonal cells, each associated to a section of the discretized SC cylinder. A sub-graph is shown in figure 1.2.4.

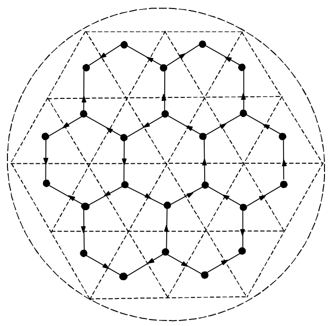

figure 1.2.4:sub-graph associated to a single section of the mesh

Figure 1.2.4 makes clear how a branch of the graph corresponds to a face of the discretization (recall that any segment of the triangular mesh slides along the axis of the cylinder to generate a rectangular face) and a node corresponds to the centre of an element. The branches that connect the nodes of a same sub-graph schematize the currents flowing in the azimuthal or radial direction, whereas the branches that connect the nodes of two different sub-graphs schematize the currents flowing in the axial direction. Moreover, an axial current can flow also through any of the faces laying on the surfaces adjacent to the electrodes, therefore the complete graph contains branches connecting all the nodes of the sub-graph relative to first section to node 73, and all the nodes of the sub-graph relative to third section to node 74.

Let us now introduce a first set of physical constrains on the unknown currents. Since the considered problem is magnetoquasistatic, the displacement current is negligible everywhere and the current density **J** is a soleinodal vector at any point. Consequently its flux through any closed surface must be zero. According with this property, the algebraic sum of the currents circulating through all the faces of any element has to be equal to zero at any instant. Moreover, the algebraic sum of the current injected or extracted from the SC domain by the generator and the currents circulating through the faces which lies on the inlet or outlet surfaces respectively, must also be always zero. These equations can be stated in a very natural way by using the incidence matrix of the oriented graph $\mathcal{G}$ associated to the mesh of the superconductor, having $(N_E + 2)$ nodes and N_C branches. In fact, the $(N_E + 2) \times N_C$ incidence matrix $[\mathbf{A}^{\mathbf{in}}]$ of the oriented graph $\mathcal{G}$ is defined as follow

- element $a_{i,j}$ is equal to +1 if branch j meets node i and points according to the outgoing direction
- element $a_{i,j}$ is equal to –1 if branch j meets node i and points according to the ingoing direction
- element $a_{i,j}$ is equal to 0 if branch j does not meet node i

Since the branches of the graph are oriented according to the currents and the nodes represent the centers of the elements (or the equipotential electrodes of the generator), the conditions of zero algebraic sum for the currents can be expressed in the following form

$$[\mathbf{A}^{\mathbf{in}}]\mathbf{I}(t) + \mathbf{a}' I_{tr}(t) = \mathbf{0} \tag{1.2.1}$$

where $\mathbf{I}(t)$ is the vector of all the N_C currents circulating through the faces the discretized superconducting domain at time t and $\mathbf{a}'$ is a vector whose *i-th* element is equal to +1 if node i represents the negative electrode of the current generator, is equal to –1 if node i represents the positive electrode and is equal to 0 otherwise. $I_{tr}(t)$ is the current of the generator at the same instant. When the current impressed by the generator is equal to zero equations (1.2.2) are homogeneous. The matrix obtained by adding (for example as last column) vector $\mathbf{a}$ to matrix $[\mathbf{A}^{\mathbf{in}}]$, is equal to the incidence matrix of the graph associated to the mesh of the SC domain and containing an additional branch representing the current generator. However, since among the (N_E + 2) rows of the incidence matrix $[\mathbf{A}^{\mathbf{in}}]$ only (N_E + 1) are independent rows, it follows that one of the equations (1.2.1) is not independent from the others. Furthermore, whatever set of (N_E + 1) rows of the incidence matrix is independent, therefore whatever of equations (1.2.1) can be eliminated to obtain a set of (N_E + 1) independent relations. We choose to eliminate the equation referring to the negative electrode of the generator (any other choice would be equivalent), that is the last row of incidence matrix, thus we write the set of the (N_E + 1) independent equations as follows

$$[\mathbf{A}]\mathbf{I}(t) = -\mathbf{a}\, I_{tr}(t) \tag{1.2.2}$$

where the $(N_E + 1) \times N_C$ matrix **[A]** and the $(N_E + 1)$ vector **a** are extracted from the incidence matrix $[\mathbf{A}^{\mathbf{in}}]$ and the vector **a'** respectively, by suppressing the last row. Equations (1.2.2) represent a first set of $(N_E + 1)$ conditions that the N_C unknowns must satisfy.

Let us carry out a step ahead and consider the electric field inside the superconducting domain. According to the $\mathbf{A} - \varphi$ formulation of magnetoquasistatics the electric field **E** at any point of the superconducting region can be expressed, at any time t, as follows

$$\mathbf{E}(\mathbf{x},t) = -\nabla\varphi(\mathbf{x},t) - \frac{\partial \mathbf{A}(\mathbf{x},t)}{\partial t} \tag{1.2.3}$$

where **A** is the magnetic vector potential and φ is the electric scalar potential. By taking the line integral of the electric field over a path connecting whatever couple of points $\mathbf{x}_h$ and $\mathbf{x}_k$ belonging to the SC domain and oriented from $\mathbf{x}_h$ to $\mathbf{x}_k$ the following equation is obtained:

$$\int_{\mathbf{x}_h}^{\mathbf{x}_k} \mathbf{E}(\mathbf{x},t)\cdot d\mathbf{x} = \varphi(\mathbf{x}_h,t) - \varphi(\mathbf{x}_k,t) - \frac{d}{dt}\int_{\mathbf{x}_h}^{\mathbf{x}_k} \mathbf{A}(\mathbf{x},t)\cdot d\mathbf{x} \tag{1.2.4}$$

The time derivative of the vector magnetic potential has been moved out of the integral because the integration path does not change with time. Moreover after the integral the space dependences disappear and the derivative becomes total. All the terms of equation (1.2.4) have the dimension of a voltage. By considering the mesh of the superconducting domain, it is possible to associate at any current flowing through a face an equation of the same type of (1.2.4). In fact, for any current flowing through a face lying inside the SC region we can chose as integration path the segment connecting centers of the two neighboring elements separated by the face, oriented according to the current, and for any current flowing through a face lying on the border with the electrodes of the generator, we can chose as integration path the segment connecting the centre of the unique element to which the face belong to and the centre of the face itself, still oriented according to the current. By expressing the same concept by means of the oriented graph $\mathcal{G}$ associated to the mesh, we can say that an equation of the same type of (1.2.4) can be associated at any branch of the graph.

We have stated since the beginning that we assume the currents flowing through the faces of the SC mesh as unknowns of the discretized problem, therefore if we mean to use equations of type (1.2.4) to solve the problem we have to express all the filed quantities there appearing as a function of the currents. To accomplish this task we first recall that the magnetic vector potential **A** at any point of the superconductor can be expressed through equation (1.1.5), which in the considered case takes the form

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int_{V_{SC}}\frac{\mathbf{J}(\mathbf{x}',t)}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'+\frac{\mu_0}{4\pi}\int_{V_{NC}}\frac{\mathbf{J}^{ext}(\mathbf{x}',t)}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}' \tag{1.2.5}$$

where the first integral represents the contribute of currents circulating over volume V_{SC} of superconducting region and the second one represents the contribute of currents circulating over volume V_{NC} of normal conducting region. If necessary the first integral of the right hand side can be calculated as sum of the integrals of the same function over all the elements of the mesh of the SC domain. The current density inside the normal conducting region is denoted with $\mathbf{J}^{ext}$. By substituting equation (1.2.5) in equation (1.2.4) we obtain

$$\begin{aligned}&\int_{\mathbf{x}_h}^{\mathbf{x}_k}\mathbf{E}(\mathbf{x},t)\cdot d\mathbf{x}=\varphi(\mathbf{x}_h,t)-\varphi(\mathbf{x}_k,t)+\\&-\frac{d}{dt}\int_{\mathbf{x}_h}^{\mathbf{x}_k}\left(\frac{\mu_0}{4\pi}\int_{V_{SC}}\frac{\mathbf{J}(\mathbf{x}',t)}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'\right)\cdot d\mathbf{x}-\frac{d}{dt}\int_{\mathbf{x}_h}^{\mathbf{x}_k}\left(\frac{\mu_0}{4\pi}\int_{V_{NC}}\frac{\mathbf{J}^{ext}(\mathbf{x}',t)}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'\right)\cdot d\mathbf{x}\end{aligned} \tag{1.2.6}$$

Equation (1.2.6) depends both on the distribution of electric field and current density inside the superconductor. However, the two vector **E** and **J** at any point are not independent. They are linked together by means of the constitutive relation of the material, which can be expressed, in the more general form, as

$$\mathbf{E}(\mathbf{x},t) = \mathbf{F}(\mathbf{J}(\mathbf{x},t),\mathbf{x},t) \tag{1.2.7}$$

For homogeneous, ohmic (linear) and time-invariant materials function **F** is linear and it is not point or time dependent. For superconductors, which are far from being linear, the constitutive relation is more complex, and depends on the temperature and the magnetic field in the considered point. However, the main features of the macroscopic behavior of superconductors are well reproduced by assuming the vectors **E** and **J** to be parallel with magnitudes following a power law [14-16], i.e.

$$\mathbf{E}(\mathbf{x},t) = E_c \left(\frac{|\mathbf{J}(\mathbf{x},t)|}{J_c(T,\mathbf{B})} \right)^{N(T,\mathbf{B})} \frac{\mathbf{J}(\mathbf{x},t)}{|\mathbf{J}(\mathbf{x},t)|} \tag{1.2.7}$$

where E_c is a conventional value, usually assumed equal to 1 μV/cm, $J_c(T,B)$ is the temperature and magnetic field dependent critical current density of the superconductor, that is, the value of the current density circulating in a point of the superconductor which produces a local electric field equal to E_c when the considered point have a temperature T and is subject to a magnetic flux density **B**. The exponent N, which also depends on temperature and magnetic field, is the non dimensional parameter which allows the best fitting of the experimental $E – J$ characteristic. In the following we will assume the superconducting region to be in thermal equilibrium with assigned temperature, thus neglecting the effects of the local heating. For the cases where the thermal effects become important, the present electromagnetic model should be coupled with a thermal model which allow to calculate at any time, the temperature distribution inside the entire SC domain.

By substituting the general constitutive relation (1.2.6) in equation (1.2.5) the following equation, depending only on electric potential and current density inside the superconducting and the normal conducting domains, is obtained

$$\int_{\mathbf{x}_h}^{\mathbf{x}_k} \mathbf{F}(\mathbf{J}(\mathbf{x},t),\mathbf{x},t)\cdot d\mathbf{x} = \varphi(\mathbf{x}_h,t) - \varphi(\mathbf{x}_k,t) + \\ -\frac{d}{dt}\int_{\mathbf{x}_h}^{\mathbf{x}_k}\left(\frac{\mu_0}{4\pi}\int_{V_{SC}}\frac{\mathbf{J}(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\right)\cdot d\mathbf{x} - \frac{d}{dt}\int_{\mathbf{x}_h}^{\mathbf{x}_k}\left(\frac{\mu_0}{4\pi}\int_{V_{NC}}\frac{\mathbf{J}^{ext}(\mathbf{x}',t))}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\right)\cdot d\mathbf{x} \tag{1.2.8}$$

To express equation (1.2.8) in terms of the unknowns of the problem it is necessary to state a link between the current density inside the superconductor and the currents circulating through the faces of the mesh. Let us assume the current density **J** to be an uniform vector inside any element of the SC domain. With such an assumption an actual current distribution inside the SC can only be reproduced in an approximate sense; the more the elements the finer the approximation. The possibility of having different courses of current density inside the elements, which is a very crucial point for the accuracy of the numerical results, will be considered later in section 1.7. The uniform current density is related to the currents flowing through the faces of the element. Let us consider a generic element i and let us denote with NF_i the number of its faces. Let $I_{j,i}$ be the current flowing through the j-th face of element i. Since index j goes from zero to NF_i it defines a local (relative to the element) labeling for the currents. Let $\mathbf{u}_{j,i}$, with j = 1, NF_i, be the normal unit vector of the j-th face of element i and let $S_{i,j}$ be its surface area. Physically, the uniform current density $\mathbf{J}_i$ inside the element should satisfy the system

$$\begin{cases} \mathbf{J}_i \cdot \mathbf{u}_{1,i} S_{1,i} = I_{1,i} \\ \dots \\ \mathbf{J}_i \cdot \mathbf{u}_{NF_i,i} S_{NF_i,i} = I_{NF_i,i} \end{cases} \tag{1.2.9}$$

In general such a vector does not exist because system (1.2.9) contains more equations than unknowns. However there will exist an unique vector which will minimize the error among its fluxes and the assigned set of currents outgoing or ingoing the element. This vector can be determined by finding the minimum, with respect to vector $\mathbf{J}_i$, of the following error function

$$F(\mathbf{J}_i) = \frac{1}{2}\sum_{j=1}^{NF_i}\left(\mathbf{J}_i^T \mathbf{u}_{j,i} S_{j,i} - I_{j,i}\right)^2 \qquad (1.2.10)$$

where the scalar product has been replaced by the matrix product between transpose of vector $\mathbf{J}_i$ and vector $\mathbf{u}_{j,i}$ (T denotes the transpose operator). By imposing the derivative of function F respect to $\mathbf{J}_i$ to be equal to zero and considering that the product of compatible $n\times m$ matrixes (m of left matrix is equal to n of right matrix) is associative, the following equation is obtained

$$\left[\sum_{j=1}^{NF_i} S_{j,i}^2 \mathbf{u}_{j,i}\mathbf{u}_{j,i}^T\right]\mathbf{J}_i - \sum_{j=1}^{NF_i} S_{j,i} I_{j,i}\mathbf{u}_{j,i} = \mathbf{0} \qquad (1.2.11)$$

Matrix that left multiply vector $\mathbf{J}_i$ in equation (1.2.11) is a 3 × 3 strictly positive defined matrix and hence it can always be inverted, therefore the expression of uniform current density as a function of currents is given by

$$\mathbf{J}_i = \left[\sum_{j=1}^{NF_i} S_{j,i}^2 \mathbf{u}_{j,i}\mathbf{u}_{j,i}^T\right]^{-1} \sum_{j=1}^{NF_i} S_{j,i} I_{j,i}\mathbf{u}_{j,i} \qquad (1.2.12)$$

Let us now introduce the local-global correspondence matrix $[\mathbf{C}_i^{lg}]$ for currents of element i, having NF_i rows and N_C columns. Element $c_{i\ j,h}^{lg}$ of matrix $[\mathbf{C}_i^{lg}]$ is equal to 1 if face where *h-th* current flows coincides with face *j-th* of element i and is otherwise equal to zero. If face j of element i lies on the boundary, its current is zero and, consequently, row *j-th* of matrix $[\mathbf{C}_i^{lg}]$ is made of all zeros. It follows that vector $\mathbf{I}_i(t)$ of currents through all the faces of element i at time t, is linked to vector $\mathbf{I}(t)$ of all currents through the faces of the superconducting mesh at same time by means of the following relation

$$\mathbf{I}_i(t) = [\mathbf{C}_i^{lg}]\mathbf{I}(t) \qquad (1.2.13)$$

By substituting equation (1.2.13) in equation (1.2.12) the following relation is obtained

$$\mathbf{J_i} = \left(\left[\sum_{j=1}^{NF_i} S_{j,i}^2 \mathbf{u}_{j,i} \mathbf{u}_{j,i}^T \right]^{-1} \left(\sum_{j=1}^{NF_i} S_{j,i} \mathbf{u}_{j,i} \left[\mathbf{C}_i^{lg} \right]_j \right) \right) \mathbf{I}(t) \tag{1.2.14}$$

where $[\mathbf{C}_i^{lg}]_j$ is the *j-th* row of matrix $[\mathbf{C}_i^{lg}]$. If follows that the uniform current density at any point of the superconducting domain can be expressed, at any instant t, as a function of all current through the faces of discretization in the following concise way

$$\mathbf{J}(\mathbf{x}, t) = [\mathbf{K}(\mathbf{x})] \mathbf{I}(t) \tag{1.2.15}$$

Matrix **[K(x)]**, having dimension $3 \times N_C$, is an element-wise uniform matrix, i. e. its elements are the same for all points **x** belonging to the same geometric element of discretized region SC. To determine the value of matrix **[K(x)]** at a given point **x'** is only necessary to find out the element i of the mesh to which point **x'** belong to and then calculate it as reported in equation (1.2.14). Matrix **[K(x)]** is very sparse; in fact only columns which are relative to currents flowing through the faces of the element containing point **x** are non zero. This means that the reconstruction of current density at any point is strictly local, i. e. it is only contributed by currents flowing in the vicinity of the point. In case of mesh made by tetrahedron at most four columns of matrix **[K(x)]** can be different from zero. They are less than four if the element have one ore more faces lying on the boundary of the SC domain. The non zero columns can be at most five or six for mesh made of prisms with triangular basis or parallelepipeds respectively.

For what concern the normal conducting part of the system, now on we assume it to be a normal coil made of N_{turns} turns. The current density is supposed to distributes uniformly inside any turn. With these assumptions current density $\mathbf{J}^{ext}$ at any point **x** of normal conducting region is a known quantity at any instant t and can be expressed as

$$\mathbf{J}^{ext}(\mathbf{x},t)=\mathbf{k}_{NC}(\mathbf{x})I_{coil}(t) \tag{1.2.16}$$

where $\mathbf{k}_{NC}(\mathbf{x})$ is the vector given by the ratio between the unit vector tangent to the direction of the turn at point $\mathbf{x}$ and the area of its cross section. If cases where the normal conducting part of the system is a generic bulk domain are considered, the distribution of current density is not a priori known, therefore the normal conducting region needs to be subdivided in a finite number of three dimensional elements as well, and the current density in any element has to be expressed as a function of all currents through the faces of the discretization. These currents must be treated as unknowns together with the currents of the SC region.

By substituting equations (1.2.15) and (1.2.6) in equation (1.2.16) the following relation is obtained

$$\begin{aligned}\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\ \mathbf{F}([\mathbf{K}(\mathbf{x})]\mathbf{I}(t),\mathbf{x},t) &= \varphi(\mathbf{x}_h,t)-\varphi(\mathbf{x}_k,t)+\\ &-\left[\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\left(\frac{\mu_0}{4\pi}\int_{V_{SC}}\frac{[\mathbf{K}(\mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\right)\right]\frac{d}{dt}\mathbf{I}(t)+\\ &-\left(\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\left(\frac{\mu_0}{4\pi}\int_{V_{NC}}\frac{\mathbf{k}_{NC}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\right)\right)\frac{d}{dt}I_{coil}(t)\end{aligned} \tag{1.2.17}$$

In case of homogeneous superconducting materials, by splitting the line integral from $\mathbf{x}_h$ to $\mathbf{x}_k$ in the sum of the integrals over the two segments connecting $\mathbf{x}_h$ to the centre of the face shared by the two elements and the latter to $\mathbf{x}_k$ and considering that matrix functions $[\mathbf{K}(\mathbf{x})]$ is element wise uniform, function $\mathbf{F}$ and can be moved out of the integral; the left side term can be then expressed as the product of a vector of geometrical coefficients times a nonlinear function of all currents.

Equation (1.2.17) states a non linear differential link between the currents of the SC region, the potentials of the nodes of the SC mesh, and the current circulating through the external coil. Let us denote with i the unique face of the SC mesh associated to points $\mathbf{x}_h$ and $\mathbf{x}_k$ and with $I_i(t)$ the current flowing through it at time t, moreover, only for convenience, let us rewrite equation (1.2.17) as follows

$$
\begin{aligned}
&\varphi(\mathbf{x}_h,t)-\varphi(\mathbf{x}_k,t)-\left(\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\left(\frac{\mu_0}{4\pi}\int_{V_{NC}}\frac{\mathbf{k}_{NC}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\right)\right)\frac{d}{dt}I_{coil}(t)= \\
&+\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\ \mathbf{F}([\mathbf{K}(\mathbf{x})]\mathbf{I}(t),\mathbf{x},t)+\left[\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\left(\frac{\mu_0}{4\pi}\int_{V_{SC}}\frac{[\mathbf{K}(\mathbf{x}')]}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'\right)\right]\frac{d}{dt}\mathbf{I}(t)
\end{aligned}
\tag{1.2.18}
$$

The left side of equation (1.2.18) contains the difference of the potentials of the points $\mathbf{x}_h$ and $\mathbf{x}_k$, and a voltage term, that we indicate with $v^{ext}_i(t)$, given by the product of a geometrical coefficient that we indicate with m_i^{coil}, having the dimension of an inductance, and the time derivative of the current flowing in the coil, which is a known quantity. The voltage term represented by the first right hand integral is a non linear function of all currents. Let us indicate with $v^r_i(t)$ this voltage and with $f_i(\boldsymbol{I}(t))$ the function. Indeed, since the reconstruction of the current density ($[\mathbf{K}(\mathbf{x})]\mathbf{I}(t)$) is strictly local and the integral is calculated over the segment connecting the points $\mathbf{x}_h$ and $\mathbf{x}_k$, function f_i depends only on the currents flowing through the faces of the elements which are crossed by the integration path. Finally, the rightmost voltage term is given by the sum of the product of the time derivative of all the unknown currents with N_C geometrical coefficients. Let us indicate with m_{ij} the coefficient which multiply the time derivative of the *j-th* current in the equation (1.2.18), relative to the *i-th* current. Even though the reconstruction of current density is strictly local, all this coefficients can be different from zero, because the innermost integral is calculated over the whole volume of the SC region and therefore it involves all the currents. From the above inspection of the various terms, it follows that equation (1.2.18) can be interpreted, at any time t, as the voltage balance equation of a circuit branch derived from two nodes h and k with potential $\varphi(\mathbf{x}_h,t)$ and $\varphi(\mathbf{x}_k,t)$ respectively and containing an impressed voltage generator $v^{ext}_i(t)$, a non linear current-controlled voltage generator $v^r_i(t)$, and a inductor coupled with $(N_C - 1)$ others. A picture of this circuit branch is shown in figure 1.2.5.

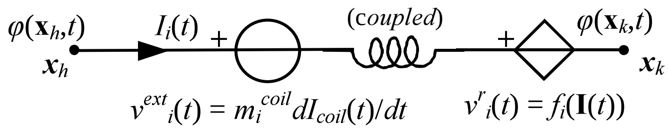

figure 1.2.5:circuit branch associated to equation (1.2.18)

By using the symbols of the figure, equation (1.2.18), which states the voltage balance of the branch, can be rewritten as

$$\varphi(\mathbf{x}_h,t)-\varphi(\mathbf{x}_k,t)-m_i^{coil}\frac{d}{dt}I_{ext}(t)=f_i(\mathbf{I}(t))+\sum_{j=1}^{NC}m_{ij}\frac{d}{dt}I_j(t) \qquad (1.2.19)$$

We can get a deeper physical insight of equation (1.2.18) and its circuit interpretation if we consider the following example. Let us refer to a group of six elements of the mesh of figure 1.2.2, which discretizes the SC cylinder of figure 1.2.1. The elements are assembled as shown by the frontal view of figure 1.2.6, where the segments connecting the centers of neighboring elements, oriented according to the current flowing through the face, are also shown.

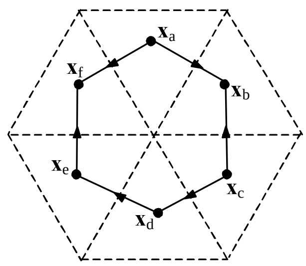

figure 1.2.6: frontal view of a group of six adjacent elements

By calculating, according to the clockwise direction, the line integral of the electric field **E**, expressed by means of equation (1.2.3), over the closed loop defined by the six segments of figure1.2.6, and substituting in it the expression (1.2.5) of magnetic scalar potential the following equation is obtained

$$\oint_{abcdefa} \mathbf{E}(\mathbf{x},t)\cdot d\mathbf{x} = -\frac{d}{dt}\oint_{abcdefa}\left(\frac{\mu_0}{4\pi}\int_{V_{SC}}\frac{\mathbf{J}(\mathbf{x}',t)}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'\right)\cdot d\mathbf{x} + \\ -\frac{d}{dt}\oint_{abcdefa}\left(\frac{\mu_0}{4\pi}\int_{V_{NC}}\frac{\mathbf{J}^{ext}(\mathbf{x}',t)}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'\right)\cdot d\mathbf{x} \tag{1.2.20}$$

By substituting in equation (1.2.20) equations (1.2.7), (1.2.15) and (1.2.16), and rearranging the terms, we obtain the equation

$$-\left(\oint_{abcdefa} d\mathbf{x}^T\left(\frac{\mu_0}{4\pi}\int_{V_{NC}}\frac{\mathbf{k}_{NC}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'\right)\right)\frac{d}{dt}I_{coil}(t) = \\ \oint_{abcdefa} d\mathbf{x}^T\ \mathbf{F}([\mathbf{K}(\mathbf{x})]\mathbf{I}(t),\mathbf{x},t) + \left[\oint_{abcdefa} d\mathbf{x}^T\left(\frac{\mu_0}{4\pi}\int_{V_{SC}}\frac{[\mathbf{K}(\mathbf{x}')]}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'\right)\right]\frac{d}{dt}\mathbf{I}(t) \tag{1.2.21}$$

The left side integral represents the electromotive force induced in the loop by the time variation of the magnetic flux linked with it and sustained by the current flowing in the coil, whereas the second of the integrals of the right hand side represents the electromotive force induced in the loop by the time variation of the magnetic flux linked with it and produced by the currents flowing in superconducting region. The first integral on the right represents the voltage drop associated to the heating by Joule effect produced by the currents flowing within the elements which are crossed by the loop. If we now apply equation (1.2.18) to any of the segments of the loop and we sum or subtract these equations depending on whether the relative segment is oriented clockwise or counterclockwise respectively, we observe that the resulting equation is exactly the same as equation (1.2.21). In terms of the circuit quantities introduced with equation (1.2.19) this relation can be expressed as

$$-\left(\sum_{pq}\pm m_{pq}^{coil}\right)\frac{d}{dt}I_{coil}(t) = \sum_{pq}\pm f_{pq}(\mathbf{I}(t)) + \sum_{pq}\left(\sum_{j=1}^{NC}\pm m_{pq,j}\frac{d}{dt}I_j(t)\right) \tag{1.2.22}$$

where pq is the couple of letters ($pq = ab, bc, \ldots, fa$) which indicates the segment. Therefore, with reference to a single branch of the loop, the independent voltage generator represents the local electromotive action impressed by the time variation of the external magnetic field, and the controlled voltage generator represents the local voltage drop associated to the local heating by Joule effect. The local electromotive action induced in the branch by the time variation of magnetic field produced by the currents circulating in the superconductor are represented by means of auto mutual induction coefficients, according to classic circuit theory.

We have seen before that it is possible to obtain as many independent equations of the type of (1.2.19), as the currents circulating through the faces of the SC mesh, that is N_C. In fact any current defines an unique couple of extremes $\mathbf{x}_h$ and $\mathbf{x}_k$ of the integration path of equation (1.2.18), from which equation (1.2.19) derives. By recalling the oriented graph $\mathcal{G}$ associated to the mesh, we can state the same property by saying that an equation of the same type of (1.2.19) can be associated to each branch of the graph. It follows that the entire system, made of a superconducting domain with an assigned transport current and subject to the influence of an external magnetic field, can be schematized by means of an equivalent electric network, having N_C branches and ($N_E + 2$) nodes. The circuit picture is completed by the fact that, as we have seen before with equations 1.2.2, the algebraic sum of all currents ingoing or outgoing each node must be zero. The graph associated to the equivalent network is exactly coincident with graph $\mathcal{G}$ associated to the mesh of the SC domain. Since the electric scalar potential φ is defined apart from a constant value, it is possible to assign arbitrarily the potential of one of the nodes of the network, therefore the circuit unknowns are N_C currents and ($N_E + 1$) potentials.

All the branches of the network are current controlled branches of the type of the one represented in figure 1.2.5. The set of the N_C independent equations of the type of (1.2.19), defining the voltage balance of each branch can be written in the following concise form

$$[\mathbf{B}]\mathbf{V}(t) - \mathbf{M}^{coil}\frac{d}{dt}I_{coil}(t) = \mathsf{F}(\mathbf{I}(t)) + [\mathbf{M}]\frac{d}{dt}\mathbf{I}(t) \qquad (1.2.23)$$

where **[B]** is an $(N_E + 1) \times N_C$ matrix whose elements are all +1, 0 or –1, $\mathbf{V}(t)$ is the vector of the $(N_E + 1)$ potentials at time t of all the nodes except the one that we have chosen to assign, $\mathbf{M}^{coil}$ is the set of the N_C mutual induction coefficients of the current circulating in the external coil, $\mathsf{F}(\mathbf{I}(t))$ is the vector of the N_C non linear functions of the currents which define the controlled voltage generators, and **[M]** is the $N_C \times N_C$ matrix of auto/mutual induction coefficients. We now recall that, in writing the $(N_E + 1)$ independent incidence equations (1.2.2), we chose to eliminate, among the $(N_E + 2)$ dependent ones, the equation referring to the node representing the negative electrode of the generator. If we now decide to assign the value zero to the potential of the same node, i.e. we assume it as the reference one, it is possible to verify by direct inspection, that matrix **[B]** of equation (1.2.23) coincides with the transpose of matrix **[A]** of equation (1.2.2); therefore the solving system of the equivalent electric network, that is a set of $(N_C + N_E + 1)$ equations in the $(N_C + N_E + 1)$ unknowns represented by the currents and the potentials, can be written as

$$\begin{cases} [\mathbf{A}]\mathbf{I}(t) = \mathbf{a}\, I_{tr}(t) \\ [\mathbf{A}]^T \mathbf{V}(t) - \mathbf{M}^{coil} \dfrac{d}{dt} I_{coil}(t) = \mathsf{F}(\mathbf{I}(t)) + [\mathbf{M}]\dfrac{d}{dt}\mathbf{I}(t) \end{cases} \tag{1.2.24}$$

System (1.2.24) can be solved directly to obtain the time evolution of the potential and currents circulating inside the SC region; when the vector $\mathbf{I}(t)$ of all currents at time t has been calculated, the instantaneous distribution of current density can be reconstructed by means of equation (1.2.15). Moreover, also the instantaneous distribution of electric field can be determined by means of equation (1.2.7) and consequently, the distribution of power consumption inside the SC can be calculated, element by element, by taking the scalar product of current density times the electric field, and multiplying it times the volume of the element. However this way of proceed is not the best one in terms of calculation time and memory requirements; we will see in the next section how, by applying a tree-cotree decomposition of the equivalent circuit, it is possible to eliminate all the electric potentials from equations (1.2.24) and obtain a reduced solving system containing only $(N_C - N_E - 1)$ unknown currents.

1.3 Problems with lower dimensionality. The tree-cotree decomposition of the equivalent network and the reduced solving system

In section 1.2 we have seen how it is possible to study the full three dimensional field distribution inside a superconducting bulk by means of an equivalent electric network, whose topology and components can be defined on the basis of the mesh of the discretized problem. The circuit unknowns, i.e. the potentials of the nodes and the currents through the branches can be determined step by step by solving the system (1.2.24). However, although completely general, this approach is not the optimal one in terms of CPU requirements and calculation time because of the two following reasons. First of all, we must consider that there are problems with an intrinsically lower dimensionality, i. e. with one or more components of the current density that are not allowed to be different from zero. Since, by means of equation (1.2.15), the local component of the current density along a particular direction is due to the currents flowing locally through the faces oriented along that direction, if a component does not exist the relative currents could be omitted. This means that only a reduced number of branches must be introduced in the equivalent network. Moreover, even when the equivalent circuit is derived according to the actual dimensionality of the problem and only the minimum number of unknowns are introduced, the size of the solving system can further be reduced by applying a tree-cotree decomposition of the network.

In order to better understand this points, let us refer to the superconducting cylinder of figure (1.2.1) and consider the case of zero transport current and uniform external magnetic field, produced by a very slim and long coil oriented along the z axis and supplied by the time varying current $I_{coil}(t)$. It is possible to demonstrate that in this case, only azimuthal currents can flow inside the SC cylinder, following a change with time of the uniform external magnetic field. Let us discretize the superconducting domain by means of the mesh of figure 1.2.2. The graph of the equivalent electric network associated to the problem in the most general case is represented in figure 1.2.3. Since no current is injected in the SC cylinder by the generator, there cannot be currents flowing through the faces lying on the interface with the positive and the negative electrode, therefore we can omit all the circuit branches connecting the two electrodes (nodes 73 and 74) with the remainder of the circuit. Moreover, since any change with time of the uniform and axial directed magnetic flux density induces currents in the SC cylinder that flow in the azimuthal direction, all the faces whose unit vector is oriented along the axial and the radial direction are not crossed by any current and consequently, the relative branches can be omitted as well. However, the elimination of all these branches yields a non connected graph; since, in the considered case, the axial and the radial directions are equipotentials, we can arbitrarily connect some couple of points lying along the same radial or axial line by means of a trivial (short circuit or having only passive components) branch, thus recovering the property of connection of the graph and avoiding the existence of floating nodes in the equivalent electric network. The graph of the equivalent circuit associated to the mesh according with the actual dimensionality of the physical problem is represented in figure 1.3.1, where a labeling of the currents of all branches is also quoted.

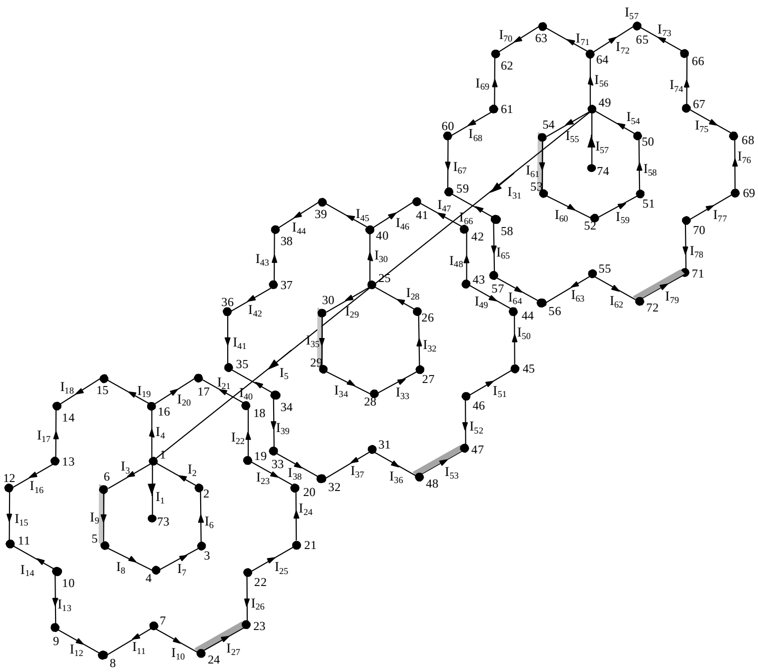

figure 1.3.1:graph associated to the 3D mesh of the SC cylinder, with only azimuthal currents

This graph contains 74 nodes, as the graph of the problem with full dimensionality (the number of the nodes does not change with the dimensionality), whereas the branches are 79 in place of 186. The reason why branches with currents I_9, I_{27}, I_{35}, I_{53}, I_{61}, and I_{79} have been pointed out will be evident in a moment. The branches connecting the nodes 1-16, 25-40, 49-64 and 73-1, 1-25, 25-49, 49-74, which refers to radial and axial currents respectively, have been maintained only to allow the graph to be connected.

The reduced electric network which arise from the graph of figure 1.3.1 contains the lowest possible number of unknowns, i.e. currents and potentials, allowed by the physics of the considered problem. The number N_C of unknown currents is equal to 79 and the number (N_E + 1) of unknown potentials is equal to 73. The circuit unknowns can be determined step by step by solving the system (1.2.24), associated to the reduced network. However this not the most convenient way to move forward; in fact, as it is well known from basic circuit theory, when a network is made only of current controlled branches, it is possible to obtain a solving system which contains only the set of the currents flowing through the branches of a whatever cotree of the graph as unknown. The equations of this reduced solving system are the voltage balance equations of the set of independent loops picked out by the cotree branches. The number N_{C_T} of branches of a tree of a given graph is equal to the total number of nodes less one. The number N_{C_C} of branches of the cotree is equal to the total number of branches of the graph less those belonging to the tree. In the case of graph of figure 1.3.1 N_{C_T} is equal to 73 and N_{C_C} is equal to 6. In order to define an algebraic algorithm which allow us to select a tree and a cotree of the graph, we start from the set of the first (N_E + 1) equations of the full solving system (1.2.24), i.e. equations (1.2.2). This equations are the Kirchhoff's current laws for all the nodes of the circuit less the reference one. They have a non homogeneous form because the superconducting bulk can be supplied by an external current generator which is not included in the equivalent network. Let us now apply to matrix **[A]** of equations (1.2.2) the following procedure

- s1. move on the left all the columns having only one element different from zero; let nc_i be the number of this columns. This step implies a rearrangement of the vector of currents

- s2. rearrange the first nc_i rows in way such they form an $nc_I \times nc_I$ identity minor which occupies the first rows and columns of matrix **[A]**. Apply to vector **a** the same rearrangement of rows carried out on matrix **[A]**

- s3. Starting from the $(nc_i + 1)$-*th*, find the next column having a non zero element on row $(nc_i + 1)$; move this column in position $(nc_i + 1)$. This step implies a rearrangement of the vector of currents

- s4. By summing or subtracting row $(nc_i + 1)$, eliminate the other non zero element from column $(nc_i + 1)_i$. Apply to vector **a** the same sum or subtraction of rows carried out on matrix **[A].** Upgrade the value of nc_i with $(nc_i + 1)$.

- s5. Repeat steps s3 and s4 until an $(N_E + 1) \times (N_E + 1)$ identity matrix is obtained on the left side of modified matrix **[A]**

Steps s1 and s2 are not essential; they could be omitted and the algebraic procedure could start from step s3, with initial value of nc_i equal to zero. However, in the case of reference node with a lot of branches connected to it (this correspond to the case of negative electrode of the generator spanning a wide surface), steps s1 and s2 make the algorithm faster.

After the above procedure is applied to equation (1.2.2), the following matrix relation is obtained

$$[\mathbf{Id}]\mathbf{I}_T(t)+[\mathbf{C}]\mathbf{I}_C(t)=-\mathbf{a}'\,I_{tr}(t) \tag{1.3.1}$$

where $\mathbf{I}_T(t)$ is the set of the first $(N_E + 1)$ components of the rearranged vector of currents, $\mathbf{I}_C(t)$ is the set of the $(N_C - N_E - 1)$ remaining ones, **[Id]** is the $(N_E + 1) \times (N_E + 1)$ identity matrix, **[C]** is an $(N_E + 1) \times (N_C - N_E - 1)$ residual matrix and **a'** is the modified vector **a**. Therefore the $(N_E + 1)$ currents of vector $\mathbf{I}_T$ can be expressed as a function of the $(N_C - N_E - 1)$ currents of vector $\mathbf{I}_C$ at any instant as follow

$$\mathbf{I}_T(t)=-\mathbf{a}'\,I_{tr}(t)-[\mathbf{C}]\mathbf{I}_C(t) \tag{1.3.2}$$

This means that the sets of branches where currents of vector $\mathbf{I}_T$ and $\mathbf{I}_C$ flow form respectively a tree and cotree of the graph to which matrix **[A]** refers to, and, consequently, the residual matrix **[C]** coincides with the matrix of the fundamental cuts associated to the tree.

By applying the above procedure to the set of equations (1.2.2) relative to the graph of figure (1.3.1), and recalling that in the considered case no current is injected in the SC cylinder by the generator, the following equations are obtained

$$\begin{bmatrix} I_{57} \\ I_{2} \\ I_{6} \\ I_{7} \\ I_{8} \\ I_{3} \\ I_{10} \\ I_{11} \\ I_{12} \\ I_{13} \\ I_{14} \\ I_{15} \\ I_{16} \\ I_{17} \\ I_{18} \\ I_{4} \\ I_{20} \\ I_{21} \\ I_{22} \\ I_{23} \\ I_{24} \\ I_{25} \\ I_{26} \\ I_{19} \\ I_{5} \\ I_{28} \\ I_{32} \\ I_{33} \\ I_{34} \\ I_{29} \\ I_{36} \\ I_{37} \\ I_{38} \\ I_{39} \\ I_{40} \\ I_{41} \\ I_{42} \\ I_{43} \\ I_{44} \\ I_{30} \\ I_{46} \\ I_{47} \\ I_{48} \\ I_{49} \\ I_{50} \\ I_{51} \\ I_{52} \\ I_{45} \\ I_{31} \\ I_{54} \\ I_{58} \\ I_{59} \\ I_{60} \\ I_{55} \\ I_{62} \\ I_{63} \\ I_{64} \\ I_{65} \\ I_{66} \\ I_{67} \\ I_{68} \\ I_{69} \\ I_{70} \\ I_{56} \\ I_{72} \\ I_{73} \\ I_{74} \\ I_{75} \\ I_{76} \\ I_{77} \\ I_{78} \\ I_{71} \\ I_{1} \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{53} \\ I_{61} \\ I_{9} \\ I_{27} \\ I_{35} \\ I_{79} \end{bmatrix} \tag{1.3.3}$$

Equations (1.3.3) show that the set of branches where currents I_9, I_{27}, I_{35}, I_{53}, I_{61}, and I_{79} flow form a cotree; this currents are pointed out in figure 1.3.1. The matrix of the fundamental cuts shows also that the current flowing in the additional branches introduced to make the graph connected (I_1, I_4, I_5, I_{30}, I_{31}, I_{56} and I_{57}), are always zero.

We have seen in the previous section that if we calculate the line integral of the electric field over a closed loop, it is possible to obtain an equation of the type of equation (1.2.22), stating the balance of the voltages acting along the loop, that involves all the currents of the equivalent electric network but does not involve any potential. If we apply this equation to all the N_{C_C} independent loops picked out by the cotree branches, and we substitute in them expression (1.3.2) of the tree currents, we obtain a reduced system made of N_{C_C} independent equations in the N_{C_C} unknowns represented by the cotree currents. In order to write the N_{C_C} independent loop equations, we can proceed in the following indirect way. Let us consider the matrix $\mathbf{[A]}^T$ appearing in the set of the last N_C equations of the full solving system (1.2.24), i.e. equations (1.2.23), and let us rewrite them in the following way

$$\mathbf{[A]}^T \mathbf{V}(t) - \mathbf{[Id]}\,\mathbf{M}^{coil}\frac{d}{dt}I_{coil}(t) = \mathbf{[Id]}\,\mathsf{F}(\mathbf{I}(t)) + \mathbf{[Id]}\,[\mathbf{M}]\frac{d}{dt}\mathbf{I}(t) \qquad (1.3.4)$$

where $\mathbf{[Id]}$ represents the $N_C \times N_C$ identity matrix. Let us now apply to equations (1.3.4) the following procedure

- s1. rearrange all the columns of matrix **[M]** by placing first those relative to the tree currents and then those relative to the cotree ones.
- s2. rearrange all the rows of matrix $\mathbf{[A]}^T$ by placing first those relative to the tree currents and then those relative to the cotree ones. Apply the same permutation of rows to matrix **[I]**.
- s3. Assign value zero to an index i.
- s4. Starting from the (i + 1)-*th*, find the next row having a non zero element on column (i + 1); move this row in position (i + 1). Apply the same exchange of rows to matrix **[I]**.
- s5. By summing or subtracting row (i + 1), eliminate the other non zero element from column (i + 1). Apply to matrix **[I]** the same sum or subtraction of rows carried out on matrix $\mathbf{[A]}^T$**.** Upgrade the value of i with (i + 1).

- s6. Repeat steps s4 and s5 until an $(N_E + 1) \times (N_E + 1)$ identity matrix is obtained on the top part of modified matrix $\mathbf{[A]}^T$

After that this procedure is applied, the following equation is obtained

$$\begin{pmatrix} \mathbf{[Id]}_{N_E+1} \\ \mathbf{[0]} \end{pmatrix} \mathbf{V}(t) - \mathbf{[Id]}' \mathbf{M}^{coil} \frac{d}{dt} I_{coil}(t) = \\ \mathbf{[Id]}' \mathsf{F}\left(\mathbf{I}_T(t), \mathbf{I}_C(t)\right) + \mathbf{[Id]}' \begin{pmatrix} [\mathbf{M}]_{11} & [\mathbf{M}]_{12} \\ [\mathbf{M}]_{21} & [\mathbf{M}]_{22} \end{pmatrix} \frac{d}{dt} \begin{pmatrix} \mathbf{I}_T(t) \\ \mathbf{I}_C(t) \end{pmatrix} \tag{1.3.5}$$

where $\mathbf{[Id]}_{N_E+1}$ represents the $(N_E + 1) \times (N_E + 1)$ identity matrix, $\mathbf{[0]}$ is an $(N_C - N_E - 1) \times (N_E + 1)$ matrix made of all zeros and matrix $\mathbf{[Id]}'$ represents the modified matrix $\mathbf{[Id]}$. Matrixes $\mathbf{[M]}_{11}$ and $\mathbf{[M]}_{12}$ are minors having $(N_E + 1)$ rows and $(N_E + 1)$ and $(N_C - N_E - 1)$ columns respectively, whereas the number of rows of minors $\mathbf{[M]}_{21}$ and $\mathbf{[M]}_{22}$ is equal to $(N_C - N_E - 1)$. Finally, by substituting equation (1.3.2) in equation (1.3.5), and considering only the last $(N_C - N_E - 1)$ rows, the following system is obtained

$$[\mathcal{M}] \frac{d}{dt} \boldsymbol{I}_C(t) = \mathcal{F}\left(\boldsymbol{I}_C(t), I_{tr}(t), I_{coil}(t)\right) \tag{1.3.6}$$

where matrix $[\mathcal{M}]$ and function $\mathcal{F}$ are defined as follow

$$[\mathcal{M}] = [\mathbf{L}] \left(\begin{pmatrix} [\mathbf{M}]_{12} \\ [\mathbf{M}]_{22} \end{pmatrix} - \begin{pmatrix} [\mathbf{M}]_{11} \\ [\mathbf{M}]_{21} \end{pmatrix} \mathbf{[C]} \right) \tag{1.3.7}$$

and

$$\mathcal{F}\left(\mathbf{I}_C(t), I_{tr}(t), I_{coil}(t)\right) = \\ [\mathbf{L}] \begin{pmatrix} [\mathbf{M}]_{11} \\ [\mathbf{M}]_{21} \end{pmatrix} \mathbf{a}' \frac{d}{dt} I_{tr}(t) - [\mathbf{L}] \mathbf{M}^{coil} \frac{d}{dt} I_{coil}(t) - [\mathbf{L}] \mathsf{F}'\left(\boldsymbol{I}_C(t), I_{tr}(t)\right) \tag{1.3.8}$$

Matrix **[L]** of the above equations is made of the last ($N_C - N_E - 1$) rows of matrix **[Id]**' and **F**' is a non linear vector function with N_C components depending only on the cotree currents and on the applied transport current. It is possible to verify by direct inspection that matrix **[L]** coincides with the matrix of the fundamental loops of the graph associated to the tree-cotree decomposition arising from equation (1.3.2).

System (1.3.6) consists of ($N_C - N_E - 1$) equations containing only the ($N_C - N_E - 1$) cotree currents as unknowns, and allows to calculate numerically their time evolution. Due to the reduced number of unknowns, the calculation time and the CPU requirements are less onerous than those of the full solving system (1.2.24). The missing ($N_E + 1$) currents circulating in the branches of the tree and, if required, the potentials of the nodes, can be determined at a later time by means of equations (1.3.2) and (1.3.5). The current density and electric field distributions inside the SC region can be calculated, at any instant, through equations (1.2.15) and (1.2.7) respectively, and finally, the resulting distribution of power losses can be calculated by taking the scalar product of current density and electric field element by element, and multiplying it with the volume.

1.4 A simple axis-symmetric problem

In order to clarify many of the concepts presented in the previous sections, let us refer to a practical example and consider a normal conducting cylinder with no transport current and subject to an uniform axial magnetic field, varying with time following a ramp with 1 *T*/*s* rate. Let the cylinder have radius r = 1*mm*, length L = 1 *m*, and electric conductivity $\sigma = 3.07 \times 10^9\ \Omega^{-1}\mathrm{m}^{-1}$ and let the uniform magnetic field be produced by a coil made of 5000 turns, arranged in two layers of 2500 each, having inner diameter d_I = 19 *mm*, outer diameter d_O = 21 *mm* and length L^{coil} = 10 *m*, oriented along the axis of the cylinder and supplied with a current changing with time following a ramp with 1592 *A*/*s* rate. Even if it is not a superconducting case, this example is convenient to understand how the numerical model works.

Let us discretize the cylinder through the mesh of figure 1.2.2, made of 72 prisms, arranged in 3 section of 24 prisms each. Even though it does not enable us to reach a good accuracy in the numerical results, such a coarse mesh allows to fully work out the numerical details and to understand all the features of the numerical model. As we have seen in section 2.3, since in the considered problem no axial and radial components of current density are allowed, the equivalent circuit will contain only the branches which schematize the currents flowing in the azimuthal direction. The graph of the equivalent circuit of the cylinder under the specified conditions is represented in figure 1.3.1. In order to calculate the time evolution of the circuit unknowns, i.e. the currents of the branches and the potentials of the nodes, all the coefficients of the solving system (1.2.24) of the equivalent network must be defined. The calculation of the incidence matrix $[\mathbf{A}]$ is straightforward. The elements of vector $\mathbf{M}^{coil}$ and matrix $[\mathbf{M}]$ and the vector function $\mathbf{F}$ can be calculated by using their definitions, provided in equation (1.2.18). In the considered case the function $\mathbf{F}$ is linear because of the linear relation among current density and electric field and can be expressed as the product of a matrix with constant elements having the dimension of a resistance and the vector of all the unknown currents. Once the solving system is completely defined the reduction algorithm presented in section 1.3 can be applied. By applying to the incidence matrix the describe procedure and considering that no transport current is injected in the cylinder, equation (1.3.3) is obtained. This equation refers to the labeling quoted in figure 1.3.1. The currents of the right side form a set of cotree currents of the equivalent circuit. By applying the second part of the reduction algorithm the following matrix **[L]** of the fundamental loops is obtained

$$
\mathbf{[L]} = \\
= \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1&1&1&-1&1&1&-1&1&1&-1&1&1&-1&1&1&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&1&1&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&-1&1&1&-1&1&1&-1&1&1&-1&1&1&-1&1&1&-1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&-1&1&1&-1&1&1&-1&1&1&-1&1&1&-1&1&1&-1&1
\end{bmatrix} \tag{1.4.1}
$$

This matrix is of course consistent with matrix **[C]** of the fundamental cuts of equation (1.3.3). As an example we note that the loop equation relative to the cotree current I_9 (third row of matrix **[L]**) is made of the sum of the voltage balance equations relative to the branches having currents I_2, I_3, I_6, I_7, I_8 and I_9 flowing through, as confirmed by the graph of figure 1.3.1. By left multiplying vector $\mathbf{M}^{coil}$ and matrix **[M]** and the vector function **F** of system (1.2.24) with matrix **[L]** and by substituting in it equation (1.3.3) the following reduce solving system is obtained:

$$
\begin{bmatrix} 9.522\text{E-}10 \\ 1.360\text{E-}10 \\ 1.360\text{E-}10 \\ 9.522\text{E-}10 \\ 1.360\text{E-}10 \\ 9.522\text{E-}10 \end{bmatrix} \frac{d}{dt} I_{coil}(t) = \begin{bmatrix} 1.016\text{E-}08\ I_{53} \\ 3.385\text{E-}09\ I_{61} \\ 3.385\text{E-}09\ I_{9} \\ 1.016\text{E-}08\ I_{27} \\ 3.385\text{E-}09\ I_{35} \\ 1.016\text{E-}08\ I_{79} \end{bmatrix} + \\
+ \begin{bmatrix} 4.687\text{E-}12 & 3.151\text{E-}18 & 3.151\text{E-}18 & 2.205\text{E-}17 & 8.184\text{E-}13 & 2.205\text{E-}17 \\ 3.151\text{E-}18 & 4.735\text{E-}13 & 3.603\text{E-}20 & 2.522\text{E-}19 & 4.501\text{E-}19 & 8.184\text{E-}13 \\ 3.151\text{E-}18 & 3.603\text{E-}20 & 4.735\text{E-}13 & 8.184\text{E-}13 & 4.501\text{E-}19 & 2.522\text{E-}19 \\ 2.205\text{E-}17 & 2.522\text{E-}19 & 8.184\text{E-}13 & 4.687\text{E-}12 & 3.151\text{E-}18 & 1.765\text{E-}18 \\ 8.184\text{E-}13 & 4.501\text{E-}19 & 4.501\text{E-}19 & 3.151\text{E-}18 & 4.735\text{E-}13 & 3.151\text{E-}18 \\ 2.205\text{E-}17 & 8.184\text{E-}13 & 2.522\text{E-}19 & 1.765\text{E-}18 & 3.151\text{E-}18 & 4.687\text{E-}12 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} I_{53} \\ I_{61} \\ I_{9} \\ I_{27} \\ I_{35} \\ I_{79} \end{bmatrix} \tag{1.4.2}
$$

By multiplying the leftmost vector with the time derivative of the current of the coil (1592 A/s in the considered case) the values of the voltages of the independent generator of any loop are obtained. These generators represent the electromotive force induced in the loops by the time variation of the magnetic flux linked with them and sustained by the current flowing in the coil. It is easy to verify that they coincide numerically with the areas of the surfaces enclosed by the loops, since the external magnetic flux density is uniform and change with time with the rate of 1 T/s. The ratio of the voltages of the outer and the inner loop is equal to seven, being equal to seven the ratio of the relative areas (see figure 1.2.4).

The first terms of the right side of equation (1.4.2) represent the voltage drop associated to the power losses which occur all along the loop; these terms are linear because the considered material is linear. In the most general case they are expressed by means of a function of more than one current and are represented by a current controlled generator. Since in the considered case, in every branch of a loop it circulates the same current, the local current density along it depends only on this current and is constant in magnitude; therefore, by calculating the dissipative terms according to their definition (see equation (1.2.18)), it results that they are expressed by means of the product of a coefficient, which represents the loop resistance, times the relative current. It is worth to note that the resistances of the loops coincides with the product of the resistivity of the material times the length of the loop, divided by the area of the cross section of the sub-domain made of the elements which are crossed by the loop. The ratio of the resistances of the outer and the inner loop is equal to three, being equal to three the ratio of the relative lengths (see figure 1.2.4 or 1.3.1).

As it can be seen from the matrix of the auto/mutual induction coefficients matrix of equation (1.4.2), the dominant inductive effect is represented by the auto inductance of any loop, together with the mutual inductance of two loops of the same section of elements. The coupling between loops of different section of elements is negligible.

Figure 1.4.1 shows the reduced equivalent circuit of the normal conducting cylinder under the specified operating conditions, associated to the solving system (1.4.2).

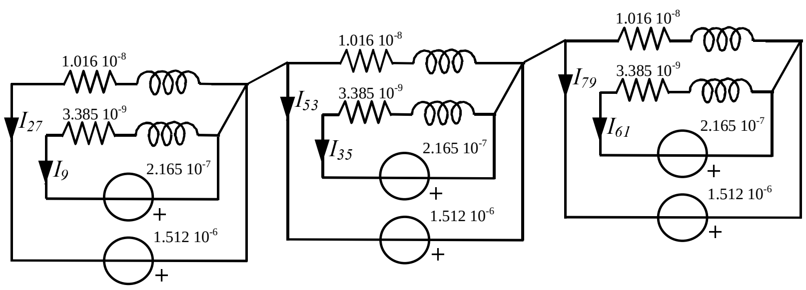

figure 1.4.1: reduced equivalent circuit of the normal conducting cylinder

The structure of the circuit reflects the structure of the mesh; in fact they can be distinguished three couple of loops, giving account of the axial distribution of the current density in the limit of the adopted discretization. The radial distribution at a given axial coordinate, still in the limit of the discretization, is given by the two corresponding loops. Since the equivalent circuit is derived under the hypothesis of axial symmetry, no azimuthal distribution arises from it.

By solving the equivalent circuit it is possible to calculate the currents circulating in the branches of the cotree, and applying afterwards equation (1.3.3), the currents circulating through all the faces of the mesh can be determined. The current density distribution can be reconstructed by means of equation (1.2.15). In the considered case a steady state distribution, independent from the axial coordinate, is obtained. Figure 1.4.2 shows a plot of the calculated current density distribution in a cross section of the cylinder.

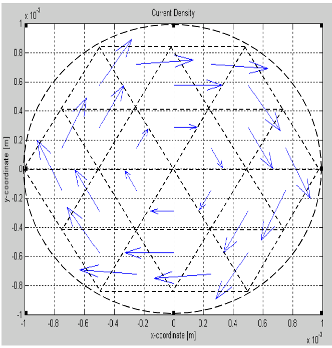

figure 1.4.2: current density distribution in a cross section of the cylinder

The magnitude of the current density is equal to 4.43E+06 A/m^2 for the innermost elements, 9.35E+06 A/m^2 for the intermediate ones, and 1.09E+07 A/m^2 for the outermost elements. The relative power losses result to be equal to 0.23, 1.02 and 1.40 mW respectively. The total power consumption inside the cylinder is equal to 72.90 mW.

It is important to note that the very simple structure of the equivalent circuit arises from the fact that, since no axial and radial components of current density were allowed in the considered problem, we have omitted since the beginning the branches which schematize the currents flowing in the axial and radial direction. By following a full 3D formulation and considering also these branches, the equivalent circuit becomes far more complex, with a very large number of unknowns. The graph of the equivalent circuit corresponding to the full 3D formulation is represented in figure 1.2.3. By solving this equivalent circuit we obtain the same solution obtained by omitting the non azimuthal branches in terms of current distribution inside the cylinder, i. e. the currents flowing in branches oriented along the axial and radial direction are found to be zero a posteriori; however the calculation time and the CPU requirements are far more onerous. In table 1.4.1 a comparison of the two equivalent circuits in terms of number of circuit quantities and solving system size is presented.

Table 1.4.1. comparison of the two equivalent circuits

	branches	*nodes*	*unknowns of the full solving system*	*unknowns of the reduced solving system*
Full 3D formulation	186	74	259	113
Only azimuthal branches	79	74	152	6

1.5 Numerical results and analytical benchmarks

In order to validate the numerical model presented in sections 1.2 and 1.3 let us now consider the following three problems for which an analytical solution is available [31].

1.5.1 Cylindrical normal conducting wire with AC transport current

Let us consider a cylindrical normal conducting wire with length *L*, radius *R* and electrical conductivity σ, connected to an AC current generator which inject the current $I_{tr}(t) = \sqrt{2} I_{rms} \sin \omega t$ inside it. The wire is connected to the generator by means of two equipotential electrodes which span its whole front and back section. With respect to a cylindrical coordinates system (r, ϑ, z) centered in the centre of the wire and with the *z* axis oriented along its axis, if the condition $L >> R$ holds, the current density distribution inside the conductor is *z*-directed and depends only on the radial coordinate *r*, and can be expressed as follow [35]:

$$J_z(r,t) = \frac{\sqrt{2} I_{rms}}{\pi R^2} \sum_{m=1}^{\infty} \frac{\sin(\omega t) + \alpha_m \left(\cos(\omega t) - e^{-\alpha_m \omega t}\right)}{1+\alpha_m^2} \frac{J_0\left(\beta_m \frac{r}{R}\right)}{J_0(\beta_m)} \tag{1.5.1}$$

where J_0 is the zero order Bessel function of the first kind, β_m is the *m-th* root of the first order Bessel function of the first kind J_1 (i.e. $J_1(\beta_m) = 0$) and α_m is a parameter expressed by $\alpha_m = \beta_m{}^2 / \mu_0 \sigma R^2$, where μ_0 is the magnetic permeability of the vacuum. The instantaneous power loss, developed by Joule effect inside the wire, corresponding to this current distribution is given by:

$$P(t) = L \int_0^R 2\pi r dr \frac{J_z^2(r,t)}{\sigma} = \frac{2 L I_{rms}^2}{\pi R_2 \sigma} \sum_{m=1}^{\infty} \left[\frac{\sin(\omega t) + \alpha_m \left(\cos(\omega t) - e^{-\alpha_m \omega t}\right)}{1+\alpha_m^2} \right]^2 \tag{1.5.2}$$

Figure 1.5.1 shows the numerical calculated profile of current density, together with the analytical one, at instant t = 2 *ms*, inside a wire having length L = 1 *m*, radius R = 1 *mm* and electrical conductivity $\sigma = 3.07 \times 10^9$ *S/m*, in the case of transport current having I_{rms} = 10 A and f = 250 *Hz*.

Figure 1.5.2 shows the numerical and the analytical calculated time evolution of total power loss which occur inside the wire during the first two periods of the transport current.

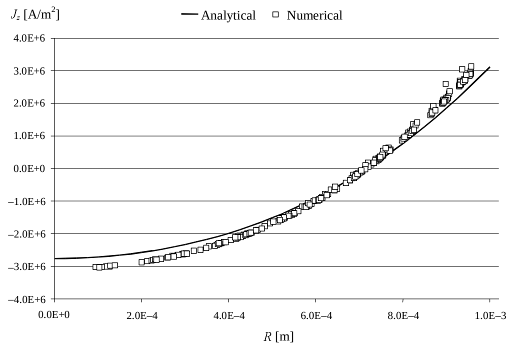

figure 1.5.1: current density distribution inside the wire at t = 2 ms

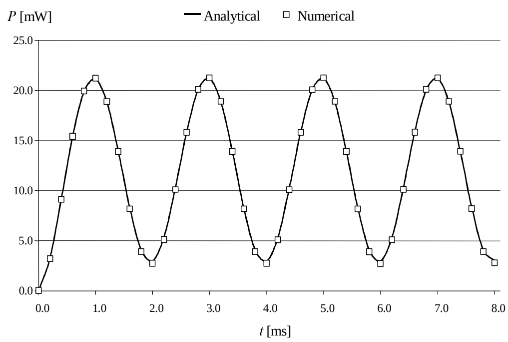

figure 1.5.2: total power losses of the wire

The mesh used for the calculation is made of 762 prisms with triangular basis, arranged in 3 section of 254 prisms each; any section covers one third of the entire length of the wire. The distributions of current density relative to the three groups of prisms are coincident, i.e. no dependence of the numerical results on the axial coordinate is observed. The numerical results obtained by means of more packed meshes are superposed to those of figure 1.51 and 1.5.2; this means that the numerical convergence is reached. The numerical and the analytical results of figure 1.51 and 1.5.2 are in a good agreement.

1.5.2 Cylindrical normal conducting wire with AC external magnetic field

Let us consider the same normal conducting wire of section 1.5.1, now having no transport current and subject to an uniform and axial oriented magnetic density, produced by an external coil and varying with time following the law $B(t)=\sqrt{2}B_{rms}\,sin(\omega t)$. With respect to a cylindrical coordinates system (*r*, ϑ, *z*) centered in the centre of the wire and with the *z* axis oriented along its axis, if the condition *L* >> *R* holds, the current density distribution inside the conductor is ϑ-directed and depends only on the radial coordinate *r*, and can be expressed as follow [35]:

$$J_{\vartheta}(r,t)=Re\left[\frac{j\sqrt{2}B_{rms}}{\mu_0}\frac{\dot{k}I_1(\dot{k}r)}{I_0(\dot{k}R_0)}e^{j\omega t}\right] \tag{1.5.3}$$

where I_0 is the zero order modified Bessel function of first kind, I_1 is the first order modified Bessel function of first kind, and $\dot{k}$ is a complex parameter expressed by $\dot{k}=(1+j)/\delta$, where the skin depth δ is defined as $\delta=\sqrt{2/\mu_0\sigma\omega}$. The instantaneous power loss, developed by Joule effect inside the wire, corresponding to this current distribution is given by:

$$P(t)=L\int_0^R 2\pi r dr\frac{J_{\vartheta}^2(r,t)}{\sigma}=\frac{2\pi L}{\sigma}\left(\frac{B_{rms}}{\mu_0}\right)^2 Re\left[S_0(\dot{k}R)-S_1(\dot{k}R)e^{j2\omega t}\right] \tag{1.5.4}$$

The complex functions S_1 and S_0 are given by $S_1(z)=\frac{1}{2}\left[\left(z\frac{I_1(z)}{I_0(z)}+1\right)^2-1-z^2\right]$ and

$S_0(z)=\frac{z^* z}{z^{*2}-z^2}\left[z^*\frac{I_1(z)}{I_0(z)}-z\frac{I_1(z^*)}{I_0(z^*)}\right]$ where star denotes the conjugate operator.

Figure 1.5.3 shows the numerical calculated profile of current density, together with the analytical one, at instant t = 9 *ms*, inside a wire having length L = 1 *m*, radius R = 1 *mm* and electrical conductivity $\sigma = 3.07 \times 10^9$ *S/m*, in the case of external magnetic flux density having B_{rms} = 1 T and f = 250 *Hz*.

Figure 1.5.4 shows the numerical and the analytical calculated time evolution of total power loss which occur inside the wire during the second and the third period of the external magnetic flux density.

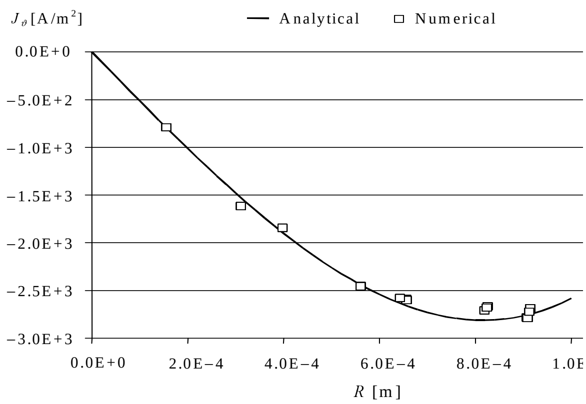

figure 1.5.2: current density distribution inside the wire t = 9 *ms*

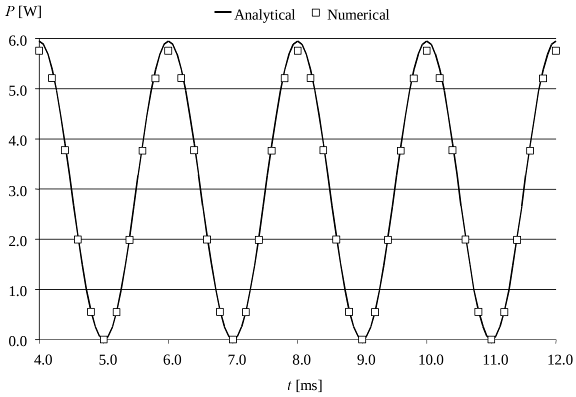

figure 1.5.4: total power losses of the wire

The mesh used for the calculation is made of 384 prisms with triangular basis, arranged in 3 section of 128 prisms each; any section covers one third of the entire length of the wire. Also in this case the distributions of current density relative to the three groups of prisms are coincident, i.e. no dependence of the numerical results on the axial coordinate is observed. The numerical results obtained by means of more packed meshes are superposed to those of figure 1.5.3 and 1.5.4; this means that the numerical convergence is reached. The numerical and the analytical results of figure 1.5.3 and 1.5.4 are in a good agreement.

1.5.3 Cylindrical superconducting wire with time-varying transport current

Let us consider a cylindrical homogeneous superconducting wire with length *L*, radius *R*, connected to a current generator which inject inside it the following time varying current

$$i_{tr}(t) = K\left[\left(1-\frac{t}{t_0}\right)^{-\frac{1}{N}} - 1\right]^{\frac{N}{N-1}} \tag{1.5.5}$$

where K and t_0 are given parameters. The wire is connected to the generator by means of two equipotentials electrodes which span its whole front and back section. Let us assume the power law as constitutive relation for the superconducting material, i.e. in any point of the wire the magnitudes of electric field and current density are related through $E = E_c\,(J/J_c)^N$, where J_c is the critical current density, E_c is a conventional value and N is a given parameter. We assume this relation not to depend on the local temperature.

With respect to a cylindrical coordinates system (r, ϑ, z) centered in the centre of the wire and with the z axis oriented along its axis the current density distribution inside the superconductor is z-directed and depends only on the radial coordinate r. It can be expressed as follow

$$J(r,t)=\begin{cases}(t_0-t)^{-\frac{1}{N}}\left[\dfrac{\mu_0 J_c^N(N-1)}{4N^2E_c}\left(\dfrac{r^2}{(t_0-t)^{1/N}}-\dfrac{R^2}{t_0^{1/N}}\right)\right]^{\frac{1}{N-1}} & ,if \quad \dfrac{r}{(t_0-t)^{1/2N}}\geq\dfrac{R}{t_0^{1/2N}}\\ 0 & ,if \quad \dfrac{r}{(t_0-t)^{1/2N}}\leq\dfrac{R}{t_0^{1/2N}}\end{cases}$$

(1.5.6)

if the conditions $L >> R$ and $t_0 = \dfrac{\mu_0 K}{4\pi N E_c}\left(\dfrac{\pi R^2 J_c}{K}\right)^N\left(1-\dfrac{1}{N}\right)^N$ hold. The instantaneous total power loss inside the wire, corresponding to this current distribution are given by:

$$P(t)=L\int_0^R 2\pi r dr\left(E_z J_z\right)=\frac{\mu_0 L}{8\pi N}\frac{I^2(t)}{t_0-t} \qquad (1.5.7)$$

Figure 1.5.5 shows the time evolution of the current supplied to the wire in the interval [0 *s* – 0.999 *s*] in the case of case of N = 6, t_0 = 1 s and K = 2218 A.

Figure 1.5.6 shows the numerical calculated current density, together with the analytical one, at instant t = 0.97 *s*, inside a superconducting wire having length L = 1 *m* and radius R = 1 *mm* supplied by the current of figure 1.5 .5. The parameters of the power law used in the calculation are $J_c = 10^9$ A/m^2, $E_c = 10^{-4}$ V/m and N = 6.

Figure 1.5.7 shows the numerical and the analytical calculated time evolution of total power loss inside the wire

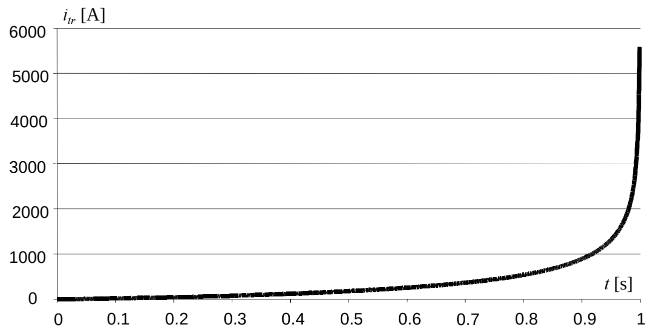

figure 1.5.5: time evolution of the current injected inside the wire

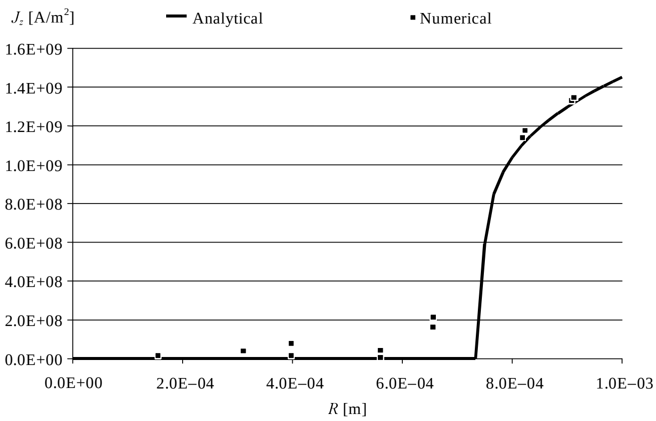

figure 1.5.6: current density distribution inside the wire at t = 0.97 s

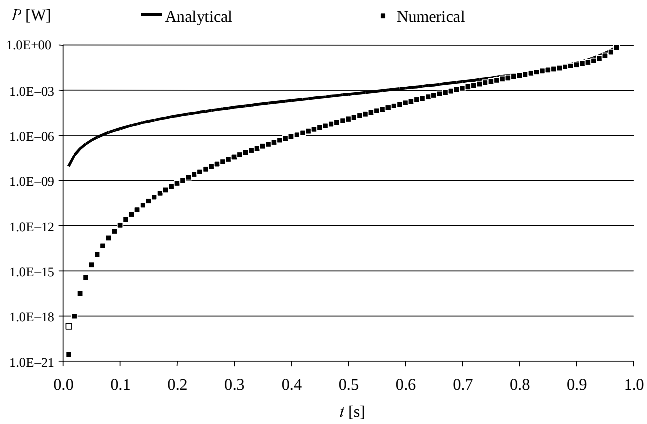

figure 1.5.7: total power losses of the wire

The mesh used for the calculation is made of 384 prisms with triangular basis, arranged in 3 section of 128 prisms each; any section covers one third of the entire length of the wire. The distributions of current density relative to the three sections of prisms are coincident, i.e. no dependence of the numerical results on the axial coordinate is observed. The numerical results obtained by means of more packed (both in the axial and in the radial direction) meshes are superposed to those of figure 1.5.6 and 1.5.7; this means that the numerical convergence is reached.

In the considered case, the current density begins to penetrate the wire from the outer part as the current injected in the wire increases; when the current injected current begin to grow very sharply, the current density propagates toward the center of the wire as a as a wave with a very sharp front. As it can be seen from figure 1.5.6, as long as the zone of the superconducting wire which is penetrated by the current is considered, the numerical calculated local current density agrees well with the analytical one. In the inner part of the wire an important disagreement is observed. For what concern the power losses, a significant discrepancy between analytical and numerical solutions at small time can be seen in figure 1.5.7. This is due to the fact that at small time the current is located in a thin surface layer, which thickness is far below the typical dimension of the cross section of the elements used. Since the model assumes an uniform current density in any element, the current is spread on an area wider than the surface layer. Therefore the numerical evaluated current density is lower than the analytical one and consequently, due to the *E*-*J* power law, the losses are underestimated. When the penetration depth becomes significant, a better fitting of analytical current distribution and losses is achieved.

1.6 Constitutive relation for non homogeneous superconductors

The model of the equivalent electric network described in the previous sections can be applied to determine the field distribution inside a superconducting domain when the constitutive relation of the material is specified at any point. However, a great part of the superconducting element used in practical applications has a composite structure made of several materials. For example, a superconducting strand is made up by a large number of filaments, with diameter of few μm, embedded in a normal conducting material matrix and twisted together as schematically reported in figure 1.6.1. Such a structure is needed for reducing ac losses and improving thermal stability. The diameter of the strand is usually in the order of the millimeter.

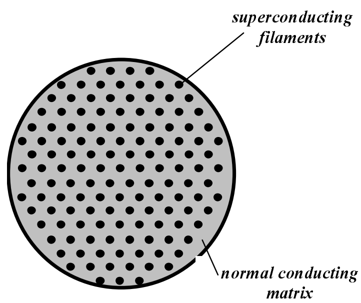

figure 1.6.1: structure of a superconducting strand

Superconducting strands are used to built up superconducting coils; when large operating currents are required, cables made of the assembling of several strands are used.

To study the current distribution inside an SC strand by means of the model of the equivalent electric network the mesh should fit exactly its structure, i.e. any elements should cover a volume filed only by an SC filament or by the normal conducting material of the matrix [36]. Due to the fact that the filament are twisted and have a diameter in the order of the μm, such an approach requires thousands of elements and full three dimensional calculation are practically impossible. However, if the detail of the field distribution is not important and only the convergence of the numerical solution with respect to some integral quantities (particularly AC losses) is required, it is possible to follow a different approach and schematize the properties of the composite material by defining a continuous (non-linear) functional dependence of the electric field on the current density which applies for any point, regardless if it lies in the SC filament or in normal matrix [7,36]. This continuous relation can be obtained by averaging the *E-J* characteristics of the SC and normal matrix material over a region which is small respect to the least dimension of the considered strand and large enough to contain a great number of SC filaments and normal matrix material areas [37]. The averaged relation is explicitly point dependent because of the twisting of the filament.

Let us consider a generic point **x** inside the strand and define the average electric field $\mathbf{E}^*(\mathbf{x})$ and the average current density $\mathbf{J}^*(\mathbf{x})$ of the point by means of the following relations:

$$\mathbf{E}^*(\mathbf{x}) = \lim_{\Delta V(\mathbf{x}) \to 0} \frac{1}{\Delta V(\mathbf{x})} \int_{\Delta V(\mathbf{x})} \mathbf{E}(\mathbf{x}') d^3\mathbf{x}' \tag{1.6.1}$$

and

$$\mathbf{J}^*(\mathbf{x}) = \lim_{\Delta V(\mathbf{x}) \to 0} \frac{1}{\Delta V(\mathbf{x})} \int_{\Delta V(\mathbf{x})} \mathbf{J}(\mathbf{x}') d^3\mathbf{x}' \tag{1.6.2}$$

where the eventual time dependence of the field quantities is implicit. The limit appearing in equations (1.6.1) and (1.6.1) has not a mathematical meaning; it only means that the considered volume is small enough, compared to the entire volume of the considered strand, for the electric field and the current density to be considered local quantities, but is still large to contain a large number of superconducting filaments embedded in the normal conducting material. Inside the volume $\Delta V(\mathbf{x})$ the SC filaments are parallel and their direction is identified by the unit vector $\mathbf{u}_l(\mathbf{x})$.

By introducing the ratio α between the volume filled by the SC filaments and the total volume of the strand, the average electric field $\mathbf{E}^*_s(\mathbf{x})$ and current density $\mathbf{J}^*_s(\mathbf{x})$ in the SC filaments lying around point **x** can be expressed as:

$$\mathbf{E}^*{}_s(\mathbf{x}) = \lim_{\Delta V(\mathbf{x})\to 0} \frac{1}{\alpha \Delta V(\mathbf{x})} \int_{\alpha \Delta V(\mathbf{x})} \mathbf{E}(\mathbf{x}') d^3\mathbf{x}' \qquad (1.6.3)$$

$$\mathbf{J}^*{}_s(\mathbf{x}) = \lim_{\Delta V(\mathbf{x})\to 0} \frac{1}{\alpha \Delta V(\mathbf{x})} \int_{\alpha \Delta V(\mathbf{x})} \mathbf{J}(\mathbf{x}') d^3\mathbf{x}' \qquad (1.6.4)$$

and, in the same way, the average electric field $\mathbf{E}^*{}_m(\mathbf{x})$ and current density $\mathbf{J}^*{}_m(\mathbf{x})$ in the normal matrix around point **x** can be expressed as:

$$\mathbf{E}^*{}_m(\mathbf{x}) = \lim_{\Delta V(\mathbf{x})\to 0} \frac{1}{(1-\alpha) \Delta V(\mathbf{x})} \int_{(1-\alpha) \Delta V(\mathbf{x})} \mathbf{E}(\mathbf{x}') d^3\mathbf{x}' \qquad (1.6.5)$$

$$\mathbf{J}^*{}_m(\mathbf{x}) = \lim_{\Delta V(\mathbf{x})\to 0} \frac{1}{(1-\alpha) \Delta V(\mathbf{x})} \int_{(1-\alpha) \Delta V(\mathbf{x})} \mathbf{J}(\mathbf{x}') d^3\mathbf{x}' \qquad (1.6.6)$$

By summing equations (1.6.3) and (1.6.5) and equations (1.6.4) and (1.6.6) it follows that

$$\mathbf{E}^*(\mathbf{x}) = \alpha \mathbf{E}^*{}_s(\mathbf{x}) + (1-\alpha)\mathbf{E}^*{}_m(\mathbf{x}) \qquad (1.6.5)$$

$$\mathbf{J}^*(\mathbf{x}) = \alpha \mathbf{J}^*{}_s(\mathbf{x}) + (1-\alpha)\mathbf{J}^*{}_m(\mathbf{x}) \qquad (1.6.6)$$

The tangential component of the electric field cannot change at the interface between the filaments and the normal matrix. Moreover, for any surface charge density is neglected, also the normal component of the current density is continuous, therefore, considering in addition equations (1.6.5) and (1.6.6), the following relations hold

$$\mathbf{E}^*{}_s(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x}) = \mathbf{E}^*{}_m(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x}) = \mathbf{E}^*(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x}) \qquad (1.6.7)$$

$$\mathbf{J}^*{}_s(\mathbf{x}) \times \mathbf{u}_t(\mathbf{x}) = \mathbf{J}^*{}_m(\mathbf{x}) \times \mathbf{u}_t(\mathbf{x}) = \mathbf{J}^*(\mathbf{x}) \times \mathbf{u}_t(\mathbf{x}) \qquad (1.6.8)$$

where **u**(**x**) is the unit vector pointing along the local direction of the filaments.

The constitute relations of the superconducting and normal conducting material are expressed by $\mathbf{E}(\mathbf{x}) = \mathbf{F}_s(\mathbf{J}(\mathbf{x}))$ and $\mathbf{E}(\mathbf{x}) = \rho_m \mathbf{J}(\mathbf{x})$ respectively, where ρ_m is the electrical resistivity of the normal material and $\mathbf{F}_s$ is an assigned function which is usually expressed by a power law. If we assume that in the vicinity of point **x** the current density **J** of the superconductor is uniform and is equal to its average value $\mathbf{J}^*{}_s$ it follows

$$\mathbf{E}^*{}_s(\mathbf{x}) = \mathbf{F}_s\left(\mathbf{J}^*{}_s(\mathbf{x})\right) \tag{1.6.9}$$

$$\mathbf{E}^*{}_m(\mathbf{x}) = \rho_m \mathbf{J}^*{}_m(\mathbf{x}) \tag{1.6.10}$$

By taking the scalar product of equation (1.6.6) with vector $\mathbf{u}_t(\mathbf{x})$ and substituting in it equations (1.6.7), (1.6.9) and (1.6.10), the following equation, relating vectors $\mathbf{J}^*(\mathbf{x})$ and $\mathbf{J}^*_s(\mathbf{x})$ is obtained

$$\mathbf{J}^*(\mathbf{x})\cdot\mathbf{u}_t(\mathbf{x}) = \alpha\mathbf{J}^*{}_s(\mathbf{x})\cdot\mathbf{u}_t(\mathbf{x}) + \frac{1-\alpha}{\rho_m}\mathbf{F}_s\left(\mathbf{J}^*{}_s(\mathbf{x})\right)\cdot\mathbf{u}_t(\mathbf{x}) \tag{1.6.11}$$

By solving equation (1.6.11) with respect to $\mathbf{J}^*_s(\mathbf{x})$ it follows

$$\mathbf{J}^*{}_s(\mathbf{x}) = \mathbf{G}^*{}_s\left(\mathbf{J}^*(\mathbf{x}),\mathbf{x}\right) \tag{1.6.12}$$

By substituting equation (1.6.12) in equation (1.6.6) we obtain

$$\mathbf{J}^*{}_m(\mathbf{x}) = \frac{1}{(1-\alpha)}\mathbf{J}^*(\mathbf{x}) - \frac{\alpha}{(1-\alpha)}\mathbf{G}^*{}_s\left(\mathbf{J}^*(\mathbf{x}),\mathbf{x}\right) = \mathbf{G}^*{}_m\left(\mathbf{J}^*(\mathbf{x}),\mathbf{x}\right) \tag{1.6.13}$$

and finally, by substituting equations (1.6.9), (1.6.12), (1.6.10) and (1.6.13) in equation (1.6.14) we obtain

$$\mathbf{E}^*(\mathbf{x}) = \alpha\mathbf{F}_s\left(\mathbf{G}^*{}_s\left(\mathbf{J}^*(\mathbf{x}),\mathbf{x}\right)\right) + \rho_m(1-\alpha)\mathbf{G}^*{}_m\left(\mathbf{J}^*(\mathbf{x}),\mathbf{x}\right) = \mathbf{F}^*\left(\mathbf{J}^*(\mathbf{x}),\mathbf{x}\right) \tag{1.6.14}$$

This equation relates the local average electric field to the local average current density and can be used as "average" constitutive relation of the superconducting strand. Its explicit dependence from the point arises from the twisting of the filaments. If we are for example interested in calculating the total power losses inside a strand subject to assigned operating conditions, it not important to know how the current density distributes over space scale in the order of few filaments diameters (tens of microns), and only the average distribution is important. Therefore a relatively non-fine discretization can be used, making the calculation time and the CPU requirements affordable, and equation (1.6.14) can be assumed as constitutive relation of the strand.

In order to see more in detail how it is possible to obtain the "average" constitutive relation, let us consider, as an example, the case of the copper stabilized niobium-titanium strand whose characteristics are listed below

diameter	0.825± 0.0025 *mm*
number of NbTi filaments	6534
diameter of the filaments	6.0 ± 0.1 μm
Cu/NbTi ratio	1.9 ÷ 2.0
SC/total volume ratio	0.34
twist pitch of the filaments	15.0 ± 1.5 *mm*
critical current at 4.2 and K, 5 T	387 A
resistivity of the copper at 4.2 K	$3.4\ 10^{-10}\ \Omega m$
RRR	> 100

This strand is produced by EUROPA METALLI S.p.A. and it will be used in the magnets of the LHC experiments, operating at a temperature of 4.2 K. Let us assume for the superconducting filaments the following constitutive relation

$$\mathbf{E}(\mathbf{x}) = E_c \left(\frac{|\mathbf{J}(\mathbf{x})|}{J_c} \right)^N \frac{\mathbf{J}(\mathbf{x})}{|\mathbf{J}(\mathbf{x})|} \tag{1.6.15}$$

where J_c is the critical current density of the NbTi at 4.2 *K* and 5 *T*, equal to $2.0\ 10^9$ A/m^2, E_c is a conventional value equal to 1 $\mu V/cm$, and the exponent N is equal to 20. The assumption of this material characteristic makes precautionary the calculation if the local temperature and magnetic flux density inside the strand never exceeds 4.2 K and 5 T respectively.

Let us consider a straight piece of strand having length L and assume a Cartesian coordinate system having the z axis parallel to the axis of the strand. The inlet section lies is located on the plane with $z = 0$. The unit vector $\mathbf{u}_t(\mathbf{x})$ parallel to the direction of the filaments, in the generic point $\mathbf{x} = (x, y, z)$ can be expressed as

$$\mathbf{u}_t(\mathbf{x}) = \begin{pmatrix} -\dfrac{\omega_p y}{\sqrt{1+\left(x^2+y^2\right)\omega_p^2}} \\ \dfrac{\omega_p x}{\sqrt{1+\left(x^2+y^2\right)\omega_p^2}} \\ \dfrac{1}{\sqrt{1+\left(x^2+y^2\right)\omega_p^2}} \end{pmatrix} \tag{1.6.16}$$

where $\omega_p = \dfrac{2\pi}{P}$ and P is the filament twist pitch (P = 1.5 10^{-2} m, ω_p = 837.758 rad/m).

We now decompose the average current density $\mathbf{J}^*(\mathbf{x})$ and the current density $\mathbf{J}^*_s(\mathbf{x})$ inside the SC filaments along the directions tangential and normal to the filaments, i.e.

$$\mathbf{J}^*{}_s(\mathbf{x}) = J^*{}_{s,t}(\mathbf{x})\mathbf{u}_t(\mathbf{x}) + \mathbf{J}^*{}_{s,n}(\mathbf{x}) \tag{1.6.17}$$

and

$$\mathbf{J}^*(\mathbf{x}) = J^*{}_t(\mathbf{x})\mathbf{u}_t(\mathbf{x}) + \mathbf{J}^*{}_n(\mathbf{x}) \tag{1.6.18}$$

where

$$J^*{}_{s,t}(\mathbf{x}) = \mathbf{J}^*{}_s(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})$$

$$\mathbf{J}_{s,n}(\mathbf{x}) = \mathbf{u}_t(\mathbf{x}) \times \left(\mathbf{J}^*{}_s(\mathbf{x}) \times \mathbf{u}_t(\mathbf{x})\right)$$

$$J^*{}_t(\mathbf{x}) = \mathbf{J}^*(\mathbf{x}) \cdot \mathbf{u}_t(\mathbf{x})$$

and

$$\mathbf{J}_n(\mathbf{x}) = \mathbf{u}_t(\mathbf{x}) \times \left(\mathbf{J}^*(\mathbf{x}) \times \mathbf{u}_t(\mathbf{x})\right).$$

By substituting equations (1.6.15), (1.6.17) and (1.6.18) in equation (1.6.11) we obtain

$$\begin{aligned}&\alpha J^*{}_{s,t}(\mathbf{x})+\\&+\frac{1-\alpha}{\rho_m}E_c\left(\frac{J^*{}_{s,t}{}^2(\mathbf{x})+\left|\mathbf{J}^*{}_{s,n}(\mathbf{x})\right|^2}{J_c{}^2}\right)^{\frac{N}{2}}\frac{J^*{}_{s,t}(\mathbf{x})}{\sqrt{J^*{}_{s,t}{}^2(\mathbf{x})+\left|\mathbf{J}^*{}_{s,n}(\mathbf{x})\right|^2}}=J^*{}_t(\mathbf{x})\end{aligned} \tag{1.6.19}$$

moreover, by substituting equations (1.6.17) and (1.6.18) in equation (1.6.8) it follows

$$\mathbf{J}^*{}_{s,n}(\mathbf{x})\times\mathbf{u}_t(\mathbf{x})=\mathbf{J}^*{}_n(\mathbf{x})\times\mathbf{u}_t(\mathbf{x}) \tag{1.6.20}$$

Equation (1.6.19) and (1.6.20) allow to determine the components of $\mathbf{J}^*{}_s(\mathbf{x})$ as a function of the components of $\mathbf{J}^*(\mathbf{x})$.

Let us assume that, if different from zero, the component of the current density along the direction normal to the filaments is locally uniform, i. e., in that direction the strand behaves like an homogeneous material. Under this hypothesis the average current of the superconductor can be expressed as

$$\mathbf{J}^*{}_s(\mathbf{x})=J^*{}_{s,t}(\mathbf{x})\mathbf{u}_t(\mathbf{x})+\mathbf{J}^*{}_n(\mathbf{x}) \tag{1.6.21}$$

With this assumption equation (1.6.20) is satisfied for every value of $J^*{}_{s,t}(\mathbf{x})$. By substituting equation (1.6.21) in equation (1.6.19) we obtain

$$J^*{}_{s,t}(\mathbf{x})\left[\alpha+\frac{1-\alpha}{\rho_m J_c}E_c\left(\frac{J^*{}_{s,t}{}^2(\mathbf{x})+\left|\mathbf{J}^*{}_n(\mathbf{x})\right|^2}{J_c{}^2}\right)^{\frac{N-1}{2}}\right]=J^*{}_t(\mathbf{x}) \tag{1.6.22}$$

which can be solved numerically to find the value of $J^*{}_{s,t}(\mathbf{x})$. If we take the derivative of equation (1.6.22) with respect to the unknown $J^*{}_{s,t}(\mathbf{x})$ we obtain

$$\left[\alpha+\frac{1-\alpha}{\rho_m J_c}E_c\left(\frac{J^*{}_{s,t}{}^2(\mathbf{x})+\left|\mathbf{J}^*{}_n(\mathbf{x})\right|^2}{J_c{}^2}\right)^{\frac{N-1}{2}}\right]+$$
$$\frac{1-\alpha}{\rho_m J_c{}^3}E_c(N-1)J^*{}_{s,t}{}^2\left(\frac{J^*{}_{s,t}{}^2(\mathbf{x})+\left|\mathbf{J}^*{}_n(\mathbf{x})\right|^2}{J_c{}^2}\right)^{\frac{N-3}{2}}=\frac{d}{J^*{}_{s,t}(\mathbf{x})}J^*{}_t(\mathbf{x}) \quad (1.6.23)$$

As it can be seen from equation (1.6.23), as long as the condition N > 3 hold, the value of of $J^*_{s,t}(\mathbf{x})$ increases monotonically with $J^*_t(\mathbf{x})$, therefore equation (1.6.22) admit always an unique solution. Once the value of $J^*_{s,t}$ has been determined by solving numerically equation (1.6.22), the value of $\mathbf{J}^*_m(\mathbf{x})$ can be calculated from equation (1.6.6), i.e.

$$\mathbf{J}^*{}_m(\mathbf{x})=\frac{1}{(1-\alpha)}\mathbf{J}^*(\mathbf{x})-\frac{\alpha}{(1-\alpha)}\mathbf{J}^*{}_s(\mathbf{x}) \quad (1.6.24)$$

Finally, the electric field can be calculated from equation (1.6.5) as follow:

$$\mathbf{E}^*(\mathbf{x})=\alpha E_c\left(\frac{J^*{}_{s,t}{}^2(\mathbf{x})+\left|\mathbf{J}^*{}_n(\mathbf{x})\right|^2}{J_c{}^2}\right)^{\frac{N}{2}}\frac{J^*{}_{s,t}(\mathbf{x})}{\sqrt{J^*{}_{s,t}{}^2(\mathbf{x})+\left|\mathbf{J}^*{}_s(\mathbf{x})\right|^2}}+$$
$$+(1-\alpha)\rho_m\mathbf{J}^*{}_m(\mathbf{x}) \quad (1.6.25)$$

1.7 Linear reconstruction of current density

In the development of equivalent electric network as presented in section 1.2, we have made the assumption of uniform current density **J** inside any element of the discretization. The profile of vector **J** inside the considered domain is reconstructed by means of equation (1.2.15), after that the currents through the faces of the discretization are calculated, and it can be discontinuous at the interfaces between the elements. With such an assumption an actual distribution of current density can only be reproduced in an approximate sense; the finer the mesh the finer the approximation. In the following we consider the possibility of having a linear course of current density inside the elements.

A linear current density $\mathbf{J}_i^1(\mathbf{x})$ at point **x** of the generic element *i* can be expressed as

$$\mathbf{J}_i^1(\mathbf{x}) = \mathbf{J}_i^0 + [\mathbf{Q}_i]\mathbf{x} \tag{1.7.1}$$

where $\mathbf{J}_i^0$ is a constant vector and $[\mathbf{Q}_i]$ is a constant 3 × 3 matrix; therefore, to define vector $\mathbf{J}_i^1(\mathbf{x})$, 12 parameters must be specified. The possible time dependence of the linear current density is implicit.

Let us denote with NF_i the number of faces of element *i* and let $I_{j,i}$ be the current flowing through its *j*-th face. Let $\mathbf{u}_{j,i}$, with $j = 1, NF_\mathrm{i}$, be the normal unit vector of the *j*-th face of element *i* and let $S_{i,j}$ be its surface area. Let us recall that, at this stage, the set of currents through the faces of the element is not subject to any constrain and can be whatever; the mathematical condition for the physical consistency of the currents are stated later by means of the incidence matrix.

The fluxes of the linear current density through the faces of element *i* should coincide with the currents circulating through them, therefore vector $\mathbf{J}_i^1(\mathbf{x})$ should satisfy the system

$$\begin{cases} \int_{S_{1,i}} \mathbf{J}_i^1(\mathbf{x}) \cdot \mathbf{u}_{1,i}\, dS = I_{1,i} \\ \dots \\ \int_{S_{NF_i,i}} \mathbf{J}_i^1(\mathbf{x}) \cdot \mathbf{u}_{NF_i,i}\, dS = I_{NF_i,i} \end{cases} \tag{1.7.2}$$

By substituting equation (1.7.1) in (1.7.2) the following equation is obtained

$$[\mathbf{P}_i]\mathbf{J}_i^{par} = \mathbf{I}_i \tag{1.7.3}$$

where $[\mathbf{P}_i]$ is an $(NF_i \times 12)$ matrix, $\mathbf{J}_i^{par}$ is the vector of the 12 parameters of the linear current density obtained by stacking vector $\mathbf{J}_i^0$ of equation (1.7.1) with the columns of matrix $[\mathbf{Q}_i]$, and $\mathbf{I}_i$ is the vector of all currents trough the faces of element *i*. The *j-th* row $\mathbf{p}_{j,i}$ of matrix $[\mathbf{P}_i]$ is defined as follow

$$\mathbf{p}_{j,i} = S_{j,i}\left[\mathbf{u}_{j,i}^T \quad x_{g\,j,i}\mathbf{u}_{j,i}^T \quad y_{g\,j,i}\mathbf{u}_{j,i}^T \quad z_{g\,j,i}\mathbf{u}_{j,i}^T\right] \tag{1.7.4}$$

where $x_{g\,j,i}$, $y_{g\,j,i}$ and $z_{g\,j,i}$ are the coordinates of the barycenter $\mathbf{x}_{g\,j,i}$ of face *j-th* face of element *i*.

For determining vector $\mathbf{J}_i^{par}$ from equation 1.7.3, matrix $[\mathbf{P}_i]$ should have 12 rows, i.e. it should refer to an element having12 faces. However matrix $[\mathbf{P}_i]$ depends on the shape of the three-dimensional element, and there can be some cases of element with twelve faces for which it is singular[2]. Therefore, in order to define a general method of linear reconstruction, which is independent from the shape of the element, a different approach has to be followed. Let us restrict ourselves to the case in which the number of currents flowing through the faces of an element is greater than 12. This condition could be reached by discretizing the domain of investigation through elements having more than 12 faces; however, since to built up such a mesh is not an easy, it is

[2] With reference to the two dimensional case it is easy to verify that the 6×6 matrix $[\mathbf{P}]$ corresponding to a regular hexagon is singular

convenient to refer still to simple elements (tetrahedron, prisms with triangular basis, parallelepipeds) and to introduce a further subdivisions of their faces. For example, by considering a prism with triangular basis and by subdividing, as shown in figure (1.7.1), its three rectangular faces in four sub-faces, an element with 14 currents flowing through is obtained.

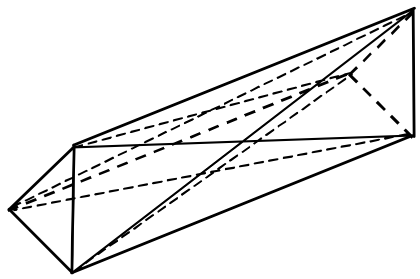

figure 1.7.1: a triangular-based prism with 14 currents

Henceforth we denote with NF_{i} the number of sub-faces of element *i* and with $I_{j,i}$ the current flowing through its *j*-th sub-face. Moreover we denote with N_C the total number of currents of the discretization, each circulating in a sub-face which does not lie on the boundary.

With reference to an element with more than 12 currents, system (1.7.3) do not admit solution for it contains more equations than unknowns. However a solution can be looked for in an approximate sense [38]; in fact an unique vector of parameters which minimizes the error among the fluxes of the corresponding linear current density and the assigned set of currents of the element exists and it can be determined by finding the minimum, with respect to vector $\mathbf{J}_i^{par}$, of the following error function

$$F\left(\mathbf{J}_i^{par}\right) = \frac{1}{2}\left\|\left[\mathbf{P}_i\right]\mathbf{J}_i^{par} - \mathbf{I}_i\right\|^2 \qquad (1.7.5)$$

By imposing the derivative of function F respect to $\mathbf{J}_i^{par}$ to be equal to zero the following equation is obtained

$$\left[[\mathbf{P}_i]^T[\mathbf{P}_i]\right]\mathbf{J}_i^{par} - [\mathbf{P}_i]^T\mathbf{I}_i = \mathbf{0} \tag{1.7.6}$$

Since the number of rows of matrix $[\mathbf{P}_i]$ is greater of the number of its columns, matrix $\left[[\mathbf{P}_i]^T[\mathbf{P}_i]\right]$ of equation (1.7.6) is always regular, therefore the unknown vector of the parameters can be calculated as

$$\mathbf{J}_i^{par} = \left[[\mathbf{P}_i]^T[\mathbf{P}_i]\right]^{-1}[\mathbf{P}_i]^T\mathbf{I}_i \tag{1.7.7}$$

By substituting equation (1.7.7) in equation (1.7.1), the following link between the linear current density of element i and the currents through its sub-faces is obtained

$$\mathbf{J}_i^1(\mathbf{x}) = \left[\mathbf{N}_i^{1-3}\right]\mathbf{I}_i + \left(\left[\mathbf{N}_i^{4-6}\right]\mathbf{I}_i, \left[\mathbf{N}_i^{7-9}\right]\mathbf{I}_i, \left[\mathbf{N}_i^{10-12}\right]\mathbf{I}_i\right)\mathbf{x} \tag{1.7.8}$$

Equation (1.7.8) is obtained by recalling that vector $\mathbf{J}_i^{par}$ is made of the stacking of vector $\mathbf{J}_i^0$ with the columns of matrix $[\mathbf{Q}_i]$, therefore matrix $\left[\mathbf{N}_i^{h-k}\right]$ consists of the group of three rows of matrix $\left[[\mathbf{P}_i]^T[\mathbf{P}_i]\right]^{-1}[\mathbf{P}_i]^T$ from the *h-th* to the *k-th*.

By introducing the ($NF_i \times NF_i$) identity matrix $[\mathbf{Id}]_{NF_i \times NF_i}$, where NF_i is now the number of sub-faces of element i, equation (1.7.8) can be rewritten as

$$\begin{aligned}\mathbf{J}_i^1(\mathbf{x}) = {} & \left[\mathbf{N}_i^{1-3}\right]\mathbf{I}_i + \\ & + \left(\left[\mathbf{N}_i^{4-6}\right]\left[\mathbf{N}_i^{7-9}\right]\left[\mathbf{N}_i^{10-12}\right]\right)\begin{pmatrix}[\mathbf{Id}]_{NF_i\times NF_i}\mathbf{I}_i & [\mathbf{Id}]_{NF_i\times NF_i}\mathbf{I}_i & [\mathbf{Id}]_{NF_i\times NF_i}\mathbf{I}_i \\ [\mathbf{Id}]_{NF_i\times NF_i}\mathbf{I}_i & [\mathbf{Id}]_{NF_i\times NF_i}\mathbf{I}_i & [\mathbf{Id}]_{NF_i\times NF_i}\mathbf{I}_i \\ [\mathbf{Id}]_{NF_i\times NF_i}\mathbf{I}_i & [\mathbf{Id}]_{NF_i\times NF_i}\mathbf{I}_i & [\mathbf{Id}]_{NF_i\times NF_i}\mathbf{I}_i\end{pmatrix}\mathbf{x}\end{aligned} \tag{1.7.9}$$

As for the case of linear reconstruction of current density, let us now introduce the local-global correspondence matrix $[\mathbf{C}_i^{lg}]$ for currents of element i, having as many rows as the number of sub-faces of element i and as many columns as the currents of the entire discretization. Element $c_i^{lg}{}_{j,h}$ of matrix $[\mathbf{C}_i^{lg}]$ is equal to 1 if sub-face where *h-th* current flows coincides with sub-face-face *j-th* of element i and is otherwise equal to zero. If sub-face j of element i lies on the boundary, its current is zero and, consequently, row *j-th* of matrix $[\mathbf{C}_i^{lg}]$ is made of all zeros. It follows that vector $\mathbf{I}_i$ of the NF_i currents through the sub-faces of element i is linked to vector $\mathbf{I}$ of the N_C currents through the sub-faces of the entire mesh by means of the following relation

$$\mathbf{I_i} = [\mathbf{C}_i^{lg}]\mathbf{I} \tag{1.7.10}$$

By substituting equation (1.7.10) in equation (1.7.9) the following relation is obtained

$$\begin{aligned}\mathbf{J}_i^1(\mathbf{x}) = &\left[\mathbf{N}_i^{1-3}\right]\mathbf{I}_i + \left(\left[\mathbf{N}_i^{4-6}\right]\left[\mathbf{N}_i^{7-9}\right]\left[\mathbf{N}_i^{10-12}\right]\right) + \\ &\begin{pmatrix} [\mathbf{Id}]_{NF_i\times NF_i}[\mathbf{C}_i^{lg}]\mathbf{I} & [\mathbf{Id}]_{NF_i\times NF_i}[\mathbf{C}_i^{lg}]\mathbf{I} & [\mathbf{Id}]_{NF_i\times NF_i}[\mathbf{C}_i^{lg}]\mathbf{I} \\ [\mathbf{Id}]_{NF_i\times NF_i}[\mathbf{C}_i^{lg}]\mathbf{I} & [\mathbf{Id}]_{NF_i\times NF_i}[\mathbf{C}_i^{lg}]\mathbf{I} & [\mathbf{Id}]_{NF_i\times NF_i}[\mathbf{C}_i^{lg}]\mathbf{I} \\ [\mathbf{Id}]_{NF_i\times NF_i}[\mathbf{C}_i^{lg}]\mathbf{I} & [\mathbf{Id}]_{NF_i\times NF_i}[\mathbf{C}_i^{lg}]\mathbf{I} & [\mathbf{Id}]_{NF_i\times NF_i}[\mathbf{C}_i^{lg}]\mathbf{I} \end{pmatrix}\mathbf{x}\end{aligned} \tag{1.7.11}$$

If follows that the linear current density at any point of the superconducting domain can be expressed as a function of all current through the sub-faces of discretization in the following concise way

$$\mathbf{J}(\mathbf{x},t) = \left[\mathbf{K}^0(\mathbf{x})\right]\mathbf{I}(t) + \left[\mathbf{K}^1(\mathbf{x})\right]\left[\mathbf{\Gamma}(\mathbf{I}(t))\right]\mathbf{x} \tag{1.7.12}$$

where the possible time dependence of linear current density and unknown currents is now explicit. Both matrixes $\left[\mathbf{K}^0(\mathbf{x})\right]$ and $\left[\mathbf{K}^1(\mathbf{x})\right]$, having dimensions (3 × N_C) and (3 × 3NF_i) respectively, are element-wise uniform matrixes, i. e. their elements are the same for all points $\mathbf{x}$ belonging to the same geometric element of discretized domain. To determine the value of matrixes $\left[\mathbf{K}^0(\mathbf{x})\right]$ and $\left[\mathbf{K}^1(\mathbf{x})\right]$ at a given point $\mathbf{x}'$ is only necessary to find out the element i of the mesh to which point $\mathbf{x}'$ belongs to and then to calculate them as reported in equation (1.7.11). Matrix $\left[\mathbf{\Gamma}(\mathbf{I}(t))\right]$ has dimension ($3NF_i$ × 3) and it depends on all the unknown currents of the discretization. Matrix $\left[\mathbf{K}^0(\mathbf{x})\right]$ is very sparse; in fact only columns which are relative to currents flowing through the sub-faces of the element containing point **x** are non zero. Also matrix $\left[\mathbf{K}^1(\mathbf{x})\right]\left[\mathbf{\Gamma}(\mathbf{I}(t))\right]$ depends only on these local currents. This means that the reconstruction of current density at any point is strictly local, i. e. it is only contributed by currents flowing in its proximity.

The graph associated to the mesh of the discretized domain in case of linear reconstruction of the current density contains more branches respect to the case of uniform reconstruction because, in order to make solvable system (1.7.3), we have introduced a further subdivision of the faces of the element. Therefore the number of unknowns of the discretized problem is greater. However, also in this case a first set of (N_E +1) physical constrains on the unknown currents, where N_E is the number of elements, can be expressed by means of the incidence matrix, according to equation (1.2.2). Moreover, a further set of N_C independent equations can be stated by taking the line integral of the electric field over the paths associated at any current flowing through a sub-face of the mesh, according to equation (1.2.8). These equations involve also (N_E +1) electric scalar potential that are unknown as well. In case of linear current density, equation (1.7.12) instead of (1.2.5) has to be substituted in equation (1.2.8), together with the constitutive relation of the material (1.2.7), thus obtaining

$$\begin{aligned}
&\varphi(\mathbf{x}_h,t)-\varphi(\mathbf{x}_k,t)-\left(\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\left(\frac{\mu_0}{4\pi}\int_{V_{NC}}\frac{\mathbf{k}_{NC}(\mathbf{x}')}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'\right)\right)\frac{d}{dt}I_{coil}(t)= \\
&+\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\mathbf{F}\left(\left(\left[\mathbf{K}^0(\mathbf{x})\right]\mathbf{I}(t)+\left[\mathbf{K}^1(\mathbf{x})\right]\left[\mathbf{\Gamma}(\mathbf{I}(t))\right]\mathbf{x}\right),\mathbf{x},t\right)+ \\
&+\left[\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\left(\frac{\mu_0}{4\pi}\int_{V_{SC}}\frac{\left[\mathbf{K}^0(\mathbf{x}')\right]}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'\right)\right]\frac{d}{dt}\mathbf{I}(t)+ \\
&+\left[\int_{\mathbf{x}_h}^{\mathbf{x}_k} d\mathbf{x}^T\left(\frac{\mu_0}{4\pi}\frac{d}{dt}\int_{V_{SC}}\frac{\left[\mathbf{K}^1(\mathbf{x}')\right]\left[\mathbf{\Gamma}(\mathbf{I}(t))\right]\mathbf{x}}{\left|\mathbf{x}-\mathbf{x}'\right|}d^3\mathbf{x}'\right)\right]
\end{aligned} \tag{1.7.13}$$

Differently form the case of linear reconstruction (see section 1.2, eq. (1.2.17)), now due to the point dependence of the linear current density, the integrand function of the first right side integral cannot to be moved out; therefore this integral cannot be expressed by means of the product of a vector of geometric coefficients with a non linear function of all the unknown currents, and has to be calculated at any time step, making the calculation more computationally expensive. For what concern the adjunctive terms which appear in the auto/mutual induction coefficients, they can be calculated once and for all at the beginning since the link with the time derivative of the unknown currents is linear.