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ANALYTICAL SOLUTIONS FOR THE CURRENT DISTRIBUTION IN A RUTHERFORD CABLE WITH N STRANDS

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Abstract – **The geometrical properties of the auto/mutual induction coefficient matrix among the strands of a Rutherford cable are investigated. The solution for the general linear case is given. The comparison with a known two-strand solution is carried out. The convergence analysis is done and the regime solution is evaluated. For the non-linear case a diagonal system is obtained and solved under the assumption of "mean" resistance. Finally, the flux ramp case and the perfect conductor cases are fully described.**

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1. Introduction

Consider a square $N \times N$ circulant symmetric matrix [A], defined by the following two properties:

1- $a_{i,1} = a_{i-1,N}$, $a_{i,j} = a_{i-1,j-1}$, with i, j = 2, … , N;

2- $a_{i,j} = a_{j,i}$, with i, j = 1, … , N.

Assume that N is even, in particular N = 2p. In this case p+1 different elements exist in the matrix [A] (in fact, consider the firs row of [A], for which $a_{1,j} = a_{1,N+2-j}$ with j = 2, … , N. There exist p–1 pairs and the unpaired elements a_{11} and $a_{1,p+1}$ ha to be added). Therefore the matrix [A] can be written as follows:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1,p} & a_{1,p+1} & a_{1,p} & \cdots & a_{13} & a_{12} \\ a_{12} & a_{11} & a_{12} & \cdots & a_{1,p-1} & a_{1,p} & a_{1,p+1} & \cdots & a_{14} & a_{13} \\ a_{13} & a_{12} & a_{11} & \cdots & a_{1,p-2} & a_{1,p-1} & a_{1,p} & \cdots & a_{15} & a_{14} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{1,p} & a_{1,p-1} & a_{1,p-2} & \cdots & a_{11} & a_{12} & a_{13} & \cdots & a_{1,p} & a_{1,p+1} \\ a_{1,p+1} & a_{1,p} & a_{1,p-1} & \cdots & a_{12} & a_{11} & a_{12} & \cdots & a_{1,p-1} & a_{1,p} \\ a_{1,p} & a_{1,p+1} & a_{1,p} & \cdots & a_{13} & a_{12} & a_{11} & \cdots & a_{1,p-2} & a_{1,p-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{13} & a_{14} & a_{15} & \cdots & a_{1,p} & a_{1,p-1} & a_{1,p-2} & \cdots & a_{11} & a_{12} \\ a_{12} & a_{13} & a_{14} & \cdots & a_{1,p+1} & a_{1,p} & a_{1,p-1} & \cdots & a_{12} & a_{11} \end{bmatrix}$$

The eigenvalues of [A] can be obtained from the N-th complex roots of the unity {1}:

$$\omega^N = 1 = e^{j2k\pi} \qquad \Rightarrow \qquad \omega_k = e^{j2k\pi/N} = e^{jk\pi/p}, \text{ with } k = p, p-1, \ldots, 1, 0, -1, \ldots, -(p-1)$$

The eigenvalues are given with k = p, p–1, …, 1, 0, –1, …, –(p–1), by:

$$\lambda_k = a_{11} + a_{12}\omega_k + \ldots + a_{1,s}\omega_k^{s-1} + \ldots + a_{1N}\omega_k^{N-1} = \sum_{s=1}^{N} a_{1,s}\omega_k^{s-1} = \sum_{s=1}^{N} a_{1,s} e^{j(s-1)k\pi/p} =$$

$$= a_{11} + \sum_{s=2}^{p} a_{1,s} e^{j(s-1)k\pi/p} + a_{1,p+1} e^{jk\pi} + \sum_{s=p+2}^{2p} a_{1,s} e^{j(s-1)k\pi/p} =$$

$$= a_{11} + \sum_{s=2}^{p} a_{1,s} e^{j(s-1)k\pi/p} + a_{1,p+1}(-1)^k + \sum_{\sigma=2}^{p} a_{1,2p+2-\sigma} e^{j(2p+1-\sigma)k\pi/p} =$$

$$= a_{11} + \sum_{s=2}^{p} a_{1,s} e^{j(s-1)k\pi/p} + a_{1,p+1}(-1)^k + \sum_{\sigma=2}^{p} a_{1,\sigma} e^{j(1-\sigma)k\pi/p} =$$

$$= a_{11} + 2\sum_{s=2}^{p} a_{1,s} \cos\frac{(s-1)k\pi}{p} + a_{1,p+1}(-1)^k$$

The equation above shows that $\lambda_q = \lambda_{-q}$, with q = 1, … , p–1. Thus, p+1 different eigenvalues of [A] exist: the algebraic multiplicity of λ_0 and λ_p is equal to 1 (and therefore also their geometric multiplicity and the dimension of the corresponding subspace are equal to 1); the algebraic multiplicity of λ_q, with q = 1, … , p–1, is equal to 2. To determine the existence of the associated spectral basis it's sufficient to demonstrate that the geometric multiplicity of λ_q, with q = 1, … , p–1, is equal to 2, determining two different eigenvectors.

It is simple to show that the vector $\mathbf{b}_k$ ($\in \mathbb{C}^N$), with k = p, p–1, …, 1, 0, defined by:

$$\mathbf{b}_k^T = \{1, \omega_k, \cdots, \omega_k^{N-1}\}$$

is an eigenvector of [A] (the index T denotes the transpose operator). In fact, we have:

$$[A]\mathbf{b}_k = \begin{Bmatrix} a_{11} + a_{12}\omega_k + \ldots + a_{1,N}\omega_k^{N-1} \\ a_{1N} + a_{11}\omega_k + \ldots + a_{1,N-1}\omega_k^{N-1} \\ \vdots \\ a_{12} + a_{13}\omega_k + \ldots + a_{11}\omega_k^{N-1} \end{Bmatrix} = \begin{Bmatrix} a_{11} + a_{12}\omega_k + \ldots + a_{1,N}\omega_k^{N-1} \\ \omega_k\left[a_{1N}\omega_k^{-1} + a_{11} + \ldots + a_{1,N-1}\omega_k^{N-2}\right] \\ \vdots \\ \omega_k^{N-1}\left[a_{12}\omega_k^{-N+1} + a_{13}\omega_k^{-N+2} + \ldots + a_{11}\right] \end{Bmatrix} =$$

$$= \begin{Bmatrix} a_{11} + a_{12}\omega_k + \ldots + a_{1,N}\omega_k^{N-1} \\ \omega_k\left[a_{1N}\omega_k^{N-1} + a_{11} + \ldots + a_{1,N-1}\omega_k^{N-2}\right] \\ \vdots \\ \omega_k^{N-1}\left[a_{12}\omega_k^{1} + a_{13}\omega_k^{2} + \ldots + a_{11}\right] \end{Bmatrix} = \begin{Bmatrix} \lambda_k \\ \omega_k\lambda_k \\ \vdots \\ \omega_k^{N-1}\lambda_k \end{Bmatrix} = \lambda_k \mathbf{b}_k$$

Note that of all the $\mathbf{b}_k$ ($\in \mathbb{C}^N$), with k = p, p–1, …, 1, 0, only $\mathbf{b}_0$ and $\mathbf{b}_p$ belong to $\mathbb{R}^N$:

$$\mathbf{b}_0^T = \{1,1,\cdots,1,1\} \qquad\qquad \mathbf{b}_p^T = \{1,-1,\cdots,1,-1\}$$

Concerning $\mathbf{b}_k$, with k = p–1, …, 1, to construct a real spectral basis the following vectors are defined:

$$\mathbf{b}_q^T = \Re\{1, \omega_q, \cdots, \omega_q^{N-1}\} = \left\{1, \cos\frac{q\pi}{p}, \cdots, \cos(2p-1)\frac{q\pi}{p}\right\}$$

$$\mathbf{b}_{-q}^T = \Im\{1, \omega_q, \cdots, \omega_q^{N-1}\} = \left\{0, \sin\frac{q\pi}{p}, \cdots, \sin(2p-1)\frac{q\pi}{p}\right\}$$

with q = 1, … , p–1. The eigenvectors $\mathbf{b}_q$ and $\mathbf{b}_{-q}$, associated to λ_q, are orthogonal. In fact, defining $\mathbf{B}_q = \mathbf{b}_q + j\,\mathbf{b}_{-q}$, it can be found that:

$$\mathbf{B}_q^T\mathbf{B}_q = 1+\omega_q^2+\cdots+\omega_q^{2(N-1)} \Rightarrow \omega_q^2\mathbf{B}_q^T\mathbf{B}_q = \mathbf{B}_q^T\mathbf{B}_q + \omega_q^{2N} - 1 \Rightarrow \left(\omega_q^2-1\right)\mathbf{B}_q^T\mathbf{B}_q = 0$$

Thus, for p ≠ q ≠ 0, we get:

$$\mathbf{B}_q^T\mathbf{B}_q = 0 = \left(\mathbf{b}_q^T\mathbf{b}_q - \mathbf{b}_{-q}^T\mathbf{b}_{-q}\right) + j\left(\mathbf{b}_q^T\mathbf{b}_{-q} + \mathbf{b}_{-q}^T\mathbf{b}_q\right) \Rightarrow \begin{cases}\left\|\mathbf{b}_q\right\|^2 = \left\|\mathbf{b}_{-q}\right\|^2\\ \mathbf{b}_q^T\mathbf{b}_{-q} = 0\end{cases}$$

Finally, the real spectral basis $\mathbf{b}_k$, with k = p, p–1, …, 1, 0, –1, …, –(p–1) is orthogonal. In fact, recalling that {2}:

$$\sum_{k=0}^{N-1}\cos(ky) = \cos\left(\frac{N-1}{2}y\right)\frac{\sin\left(\frac{N}{2}y\right)}{\sin\left(\frac{1}{2}y\right)} \qquad \sum_{k=0}^{N-1}\sin(ky) = \sin\left(\frac{N-1}{2}y\right)\frac{\sin\left(\frac{N}{2}y\right)}{\sin\left(\frac{1}{2}y\right)}$$

It results that, with s ≠ q (and thus s ± q ≠ ±N):

$$\mathbf{b}_s^T\mathbf{b}_q = \sum_{k=0}^{N-1}\cos\left(k\frac{2s\pi}{N}\right)\cos\left(k\frac{2q\pi}{N}\right) = \frac{1}{2}\sum_{k=0}^{N-1}\cos\left(k\frac{2(s+q)\pi}{N}\right) + \frac{1}{2}\sum_{k=0}^{N-1}\cos\left(k\frac{2(s-q)\pi}{N}\right) =$$
$$= \frac{1}{2}\cos\left(\frac{(N-1)(s+q)\pi}{N}\right)\frac{\sin((s+q)\pi)}{\sin\left(\frac{(s+q)\pi}{N}\right)} + \frac{1}{2}\cos\left(\frac{(N-1)(s-q)\pi}{N}\right)\frac{\sin((s-q)\pi)}{\sin\left(\frac{(s-q)\pi}{N}\right)} = 0$$

$$\mathbf{b}_s^T\mathbf{b}_{-q} = \sum_{k=0}^{N-1}\cos\left(k\frac{2s\pi}{N}\right)\sin\left(k\frac{2q\pi}{N}\right) = \frac{1}{2}\sum_{k=0}^{N-1}\sin\left(k\frac{2(q+s)\pi}{N}\right) + \frac{1}{2}\sum_{k=0}^{N-1}\sin\left(k\frac{2(q-s)\pi}{N}\right) =$$
$$= \frac{1}{2}\sin\left(\frac{(N-1)(s+q)\pi}{N}\right)\frac{\sin((s+q)\pi)}{\sin\left(\frac{(s+q)\pi}{N}\right)} + \frac{1}{2}\sin\left(\frac{(N-1)(q-s)\pi}{N}\right)\frac{\sin((q-s)\pi)}{\sin\left(\frac{(q-s)\pi}{N}\right)} = 0$$

$$\mathbf{b}_{-s}^T\mathbf{b}_{-q} = \sum_{k=0}^{N-1}\sin\left(k\frac{2s\pi}{N}\right)\sin\left(k\frac{2q\pi}{N}\right) = \frac{1}{2}\sum_{k=0}^{N-1}\cos\left(k\frac{2(s-q)\pi}{N}\right) - \frac{1}{2}\sum_{k=0}^{N-1}\cos\left(k\frac{2(s+q)\pi}{N}\right) =$$
$$= \frac{1}{2}\cos\left(\frac{(N-1)(s-q)\pi}{N}\right)\frac{\sin((s-q)\pi)}{\sin\left(\frac{(s-q)\pi}{N}\right)} + \frac{1}{2}\cos\left(\frac{(N-1)(s+q)\pi}{N}\right)\frac{\sin((s+q)\pi)}{\sin\left(\frac{(s+q)\pi}{N}\right)} = 0$$

To obtain an orthonormal spectral basis, note that: 1) $\|\mathbf{b}_0\|^2 = \|\mathbf{b}_p\|^2 = N$; 2) $\|\mathbf{b}_q\|^2 + \|\mathbf{b}_{-q}\|^2 = N$; 3) $\|\mathbf{b}_q\|^2 = \|\mathbf{b}_{-q}\|^2$; and thus $\|\mathbf{b}_q\|^2 = \|\mathbf{b}_{-q}\|^2 = N/2 = p$. therefore, it's possible to define the orthonormal spectral basis, as follows:

$$\mathbf{b}_0^T = \left\{\frac{1}{\sqrt{N}},\frac{1}{\sqrt{N}},\cdots,\frac{1}{\sqrt{N}},\frac{1}{\sqrt{N}}\right\}$$

$$\mathbf{b}_p^T = \left\{\frac{1}{\sqrt{N}},\frac{-1}{\sqrt{N}},\cdots,\frac{1}{\sqrt{N}},\frac{-1}{\sqrt{N}}\right\}$$

$$\mathbf{b}_q^T = \left\{ \frac{1}{\sqrt{p}}, \frac{1}{\sqrt{p}} \cos\frac{q\pi}{p}, \cdots, \frac{1}{\sqrt{p}} \cos(2p-1)\frac{q\pi}{p} \right\}$$

$$\mathbf{b}_{-q}^T = \left\{ 0, \frac{1}{\sqrt{p}} \sin\frac{q\pi}{p}, \cdots, \frac{1}{\sqrt{p}} \sin(2p-1)\frac{q\pi}{p} \right\}$$

with q = 1, … , p–1. Since any spectral basis is complete, it's possible to decompose uniquely any given vector $\mathbf{u} \in \mathrm{R}^N$ as follows:

$$\mathbf{u} = \sum_{k=-(p-1)}^{p} \mathbf{b}_k \left(\mathbf{b}_k^T \mathbf{u} \right)$$

and thus: $[A]\mathbf{u} = \sum_{k=-(p-1)}^{p} \lambda_k \mathbf{b}_k \left(\mathbf{b}_k^T \mathbf{u} \right)$

Finally note that:

1) given a square $N \times N$ circulant symmetric positive-definite matrix [M];

2) given a square $N \times N$ circulant symmetric [G] such that $g_{11} = -\sum_{k=2}^{N} g_{1k}$;

Thus we have that:

1 - the eigenvalues of [M] are positive and given by:

$$\lambda_k = m_{11} + 2\sum_{s=2}^{p} m_{1,s} \cos\frac{(s-1)k\pi}{p} + m_{1,p+1}(-1)^k$$

with k = p, p–1, …, 1, 0, –1, …, –(p–1). Thus [M] $\mathbf{b}_k = \lambda_k \mathbf{b}_k$;

2 - the eigenvalues of [G] are negative (except for one that is null) e given by:

$$\gamma_k = -2\sum_{s=2}^{p} g_{1,s} \left[1 - \cos\left(\frac{(s-1)k\pi}{p} \right) \right] - g_{1,p+1} \left[1 - (-1)^k \right]$$

with k = p, p–1, …, 1, 0, –1, …, –(p–1) (therefore, in particular $\gamma_0 = 0$). Thus [G] $\mathbf{b}_k = \gamma_k \mathbf{b}_k$;

3 - the orthonormal spectral basis $\mathbf{b}_k$, with k = p, p–1, …, 1, 0, –1, …, –(p–1), it's the same for [M] and [G]:

$$\mathbf{b}_0^T = \left\{ \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \cdots, \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}} \right\} \qquad \mathbf{b}_p^T = \left\{ \frac{1}{\sqrt{N}}, \frac{-1}{\sqrt{N}}, \cdots, \frac{1}{\sqrt{N}}, \frac{-1}{\sqrt{N}} \right\}$$

$$\mathbf{b}_q^T = \left\{ \frac{1}{\sqrt{p}}, \frac{1}{\sqrt{p}} \cos\frac{q\pi}{p}, \cdots, \frac{1}{\sqrt{p}} \cos(2p-1)\frac{q\pi}{p} \right\} \qquad \mathbf{b}_{-q}^T = \left\{ 0, \frac{1}{\sqrt{p}} \sin\frac{q\pi}{p}, \cdots, \frac{1}{\sqrt{p}} \sin(2p-1)\frac{q\pi}{p} \right\}$$

The analysis for N odd is really equivalent, but the formulas for the eigenvalues and for the orthonormal spectral basis are different. Assuming that N = 2p + 1, it can be found that:

1) the eigenvalues of [M] are again positive and given by:

$$\lambda_k = m_{11} + 2\sum_{s=2}^{p+1} m_{1,s} \cos\frac{2(s-1)k\pi}{2p+1}$$

where the index k spans the set {p, p–1, …, 1, 0, −1, …, −p}.

2) the eigenvalues of **g** are again negative and given by:

$$\gamma_k = -2\sum_{s=2}^{p+1} g_{1,s}\left[1-\cos\left(\frac{2(s-1)k\pi}{2p+1}\right)\right]$$

where the index k spans the set {p, p–1, …, 1, 0, −1, …, −p}.

3) the orthonormal spectral basis **b** is defined as follows:

$$\mathbf{b}_0^T = \left\{\frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}, \cdots, \frac{1}{\sqrt{N}}, \frac{1}{\sqrt{N}}\right\}$$

$$\mathbf{b}_q^T = \left\{\sqrt{\frac{2}{2p+1}}, \sqrt{\frac{2}{2p+1}}\cos\frac{2q\pi}{2p+1}, \cdots, \sqrt{\frac{2}{2p+1}}\cos\left(2p\frac{2q\pi}{2p+1}\right)\right\}$$

$$\mathbf{b}_{-q}^T = \left\{0, \sqrt{\frac{2}{2p+1}}\sin\frac{2q\pi}{2p+1}, \cdots, \sqrt{\frac{2}{2p+1}}\sin\left(2p\frac{2q\pi}{2p+1}\right)\right\}$$

with q = 1, … , p.

2. The Solution for the Linear Case

Assume that the currents flowing in the N-strand Rutherford cable are described by the following parabolic system:

$$(P)\begin{cases}[G][M]\frac{\partial \mathbf{i}}{\partial t}(x,t) + [G][R]\mathbf{i}(x,t) + \frac{\partial^2 \mathbf{i}}{\partial x^2}(x,t) = [G]\mathbf{v}(x,t) \\ i_\alpha(x=0,t) = i_\alpha(x=L,t) = \frac{I(t)}{N}, \text{ con } \alpha = 1,\cdots,N \qquad \text{and } \mathbf{i}, \mathbf{v} \in \mathrm{R}^N \\ \mathbf{i}(x,t=0) = \mathbf{i}^{(0)}(x)\end{cases}$$

with [R] = r [I], where [I] is the identity matrix, r is constant and [M] and [G] are constant-coefficient matrixes with the above described properties. The boundary conditions can be synthetically written as (o):

(o) Note that projecting (P) on $\mathbf{b}_0$, it results:

$$(P_0)\begin{cases}\frac{\partial^2}{\partial x^2}\{\mathbf{b}_0^T\mathbf{i}(x,t)\} = 0 \\ \mathbf{b}_0^T\mathbf{i}(x=0,t) = \mathbf{b}_0^T\mathbf{i}(x=L,t) = \frac{I(t)}{\sqrt{N}} \Rightarrow \mathbf{b}_0^T\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{N}} \Rightarrow \mathbf{b}_0^T\mathbf{i}^{(0)}(x) = \frac{I(t=0)}{\sqrt{N}} \text{ (compatibility condition)} \\ \mathbf{b}_0^T\mathbf{i}(x,t=0) = \mathbf{b}_0^T\mathbf{i}^{(0)}(x)\end{cases}$$

$$\mathbf{i}(x=0,t)=\mathbf{i}(x=L,t)=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0$$

Defining the current variations as:

$$\delta\mathbf{i}(x,t)=\mathbf{i}(x,t)-\frac{I(t)}{\sqrt{N}}\mathbf{b}_0$$

the problem can be restated as (note that [G] [M] $\mathbf{b}_0$ = [G] λ_0 $\mathbf{b}_0$ = γ_0 λ_0 $\mathbf{b}_0$ = 0):

$$(P')\begin{cases}[G][M]\dfrac{\partial\delta\mathbf{i}}{\partial t}(x,t)+[G]r\delta\mathbf{i}(x,t)+\dfrac{\partial^2\delta\mathbf{i}}{\partial x^2}(x,t)=[G]\mathbf{v}(x,t)\\ \delta\mathbf{i}(x=0,t)=\delta\mathbf{i}(x=L,t)=0\\ \delta\mathbf{i}(x,t=0)=\mathbf{i}^{(0)}(x)-\dfrac{I(t=0)}{\sqrt{N}}\mathbf{b}_0\end{cases}$$

Utilizing the trigonometric basis, orthogonal in [0, L], $\{\sin(n\pi x/L)\}_n$, with $n\in\mathbb{N}$, the following series are defined:

$$\mathbf{F}(x,t)=\sum_{n=1}^{\infty}\mathbf{F}_n(t)\sin\left(\frac{n\pi x}{L}\right)\Leftrightarrow\mathbf{F}_n(t)=\frac{2}{L}\int_0^L\mathbf{F}(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)d\xi.$$

and then the problem can be restated as (note that the boundary conditions are essential to obtain $\frac{2}{L}\int_0^L\frac{\partial^2\delta\mathbf{i}}{\partial x^2}(\xi,t)\sin\left(\frac{n\pi\xi}{L}\right)d\xi=-\left(\frac{n\pi}{L}\right)^2\delta\mathbf{i}_n$):

$$(P'_n)\begin{cases}[G][M]\dfrac{d\delta\mathbf{i}_n}{dt}(t)+[G]r\delta\mathbf{i}_n(t)-\left(\dfrac{n\pi}{L}\right)^2\delta\mathbf{i}_n(t)=[G]\mathbf{v}_n(t)\\ \delta\mathbf{i}_n(t=0)=\mathbf{i}_n^{(0)}-\dfrac{I(t=0)}{\sqrt{N}}\mathbf{b}_0\dfrac{2}{n\pi}\left[1-(-1)^n\right]\end{cases}\text{, with } n\in\mathbb{N}$$

Utilizing the orthonormal spectral basis $\mathbf{b}_k$, with k = p, p–1, …, 1, 0, –1, … –(p–1), defined above, we get:

$$\delta\mathbf{i}_n(t)=\sum_{k=-(p-1)}^{p}\mathbf{b}_k\eta_{n,k}(t)\Leftrightarrow\eta_{n,k}(t)=\mathbf{b}_k^T\delta\mathbf{i}_n(t)\qquad\mathbf{v}_n(t)=\sum_{k=-(p-1)}^{p}\mathbf{b}_k\nu_{n,k}(t)\Leftrightarrow\nu_{n,k}(t)=\mathbf{b}_k^T\mathbf{v}_n(t)$$

and then the problem can be restated as (with $n\in\mathbb{N}$):

$$(P'_{n,k})\begin{cases}\gamma_k\lambda_k\dfrac{d\eta_{n,k}}{dt}(t)+\gamma_k r\eta_{n,k}(t)-\left(\dfrac{n\pi}{L}\right)^2\eta_{n,k}(t)=\gamma_k\nu_{n,k}(t)\\ \eta_{n,k}(t=0)=\mathbf{b}_k^T\mathbf{i}_n^{(0)}\end{cases}\text{, with } k = p, p-1, \dots, 1, -1, \dots -(p-1)$$

Note that k ≠ 0, in fact $\eta_{n,0}=0$ as a consequence of $\mathbf{b}_0^T\delta\mathbf{i}(x,t)=0$. The problem ($P'_{n,k}$) is directly solvable:

$$\begin{cases}\lambda_k\dfrac{d\eta_{n,k}}{dt}+\left[r-\dfrac{1}{\gamma_k}\left(\dfrac{n\pi}{L}\right)^2\right]\eta_{n,k}=\nu_{n,k}(t)\\ \eta_{n,k}(t=0)=\mathbf{b}_k^T\mathbf{i}_n^{(0)}\end{cases}\Rightarrow$$

$$\Rightarrow \eta_{n,k}(t) = \left(\mathbf{b}_k^T \mathbf{i}_n^{(0)}\right) e^{\frac{-t}{\lambda_k}\left[r - \frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]} + \frac{1}{\lambda_k}\int_0^t \nu_{n,k}(\tau)\, e^{\frac{-(t-\tau)}{\lambda_k}\left[r - \frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]} d\tau$$

Reconstructing the solution, we get:

$$\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{N}}\mathbf{b}_0 + \sum_{\substack{k=-(p-1)\\k\neq 0}}^{p} \mathbf{b}_k \sum_{n=1}^{\infty} \eta_{n,k}(t) \sin\left(\frac{n\pi x}{L}\right) \Rightarrow$$

$$\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{N}}\mathbf{b}_0 + \sum_{\substack{k=-(p-1)\\k\neq 0}}^{p} \mathbf{b}_k \sum_{n=1}^{\infty} \left(\mathbf{b}_k^T \mathbf{i}_n^{(0)}\right) e^{\frac{-t}{\lambda_k}\left[r - \frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]} \sin\left(\frac{n\pi x}{L}\right) +$$

$$\Rightarrow$$

$$+ \int_0^t d\tau \sum_{\substack{k=-(p-1)\\k\neq 0}}^{p} \frac{\mathbf{b}_k}{\lambda_k} \sum_{n=1}^{\infty} \left(\mathbf{b}_k^T \mathbf{v}_n(\tau)\right) e^{\frac{-(t-\tau)}{\lambda_k}\left[r - \frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]} \sin\left(\frac{n\pi x}{L}\right)$$

$$\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{N}}\mathbf{b}_0 + \frac{2}{L}\int_0^L d\xi \sum_{\substack{k=-(p-1)\\k\neq 0}}^{p} \mathbf{b}_k \left(\mathbf{b}_k^T \mathbf{i}^{(0)}(\xi)\right) \sum_{n=1}^{\infty} e^{\frac{-t}{\lambda_k}\left[r - \frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi \xi}{L}\right) +$$

$$\Rightarrow$$

$$+ \frac{2}{L}\int_0^L d\xi \int_0^t d\tau \sum_{\substack{k=-(p-1)\\k\neq 0}}^{p} \frac{\mathbf{b}_k}{\lambda_k} \left(\mathbf{b}_k^T \mathbf{v}(\xi,\tau)\right) \sum_{n=1}^{\infty} e^{\frac{-(t-\tau)}{\lambda_k}\left[r - \frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi \xi}{L}\right)$$

$$\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{N}}\mathbf{b}_0 + \frac{2}{L}\int_0^L d\xi \sum_{\substack{k=-(p-1)\\k\neq 0}}^{p} \mathbf{b}_k \left(\mathbf{b}_k^T \mathbf{i}^{(0)}(\xi)\right) \Gamma_k(x,\xi,t) +$$

$$+ \frac{2}{L}\int_0^L d\xi \int_0^t d\tau \sum_{\substack{k=-(p-1)\\k\neq 0}}^{p} \frac{\mathbf{b}_k}{\lambda_k} \left(\mathbf{b}_k^T \mathbf{v}(\xi,\tau)\right) \Gamma_k(x,\xi,t-\tau)$$

Recalling that $\sin(\omega_1)\sin(\omega_2) = \frac{1}{2}\left[\cos(\omega_1 - \omega_2) - \cos(\omega_1 + \omega_2)\right]$ and the definition of the elliptic theta function ϑ_3 {3}(oo): $\vartheta_3(u,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nu$, the Green functions can be written as (ooo):

(oo) the fundamental propriety of the function ϑ_3 (u, q) is: $q\frac{\partial \vartheta_3}{\partial q}(u,q) = -\frac{1}{4}\frac{\partial^2 \vartheta_3}{\partial u^2}(u,q)$

(ooo) Note that: $\gamma_k \lambda_k \frac{\partial \Gamma_k}{\partial t}(x,\xi,t) + \gamma_k r \Gamma_k(x,\xi,t) + \frac{\partial^2 \Gamma_k}{\partial x^2}(x,\xi,t) = 0$, with k = p, p−1, …, 1, −1, … −(p−1).

$$\Gamma_k(x,\xi;t)=\sum_{n=1}^{\infty} e^{\frac{-t}{\lambda_k}\left[r-\frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)=$$

$$=\frac{e^{\frac{-rt}{\lambda_k}}}{2}\sum_{n=1}^{\infty} e^{\frac{tn^2}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\cos\left(\frac{n\pi(x-\xi)}{L}\right)-\frac{e^{\frac{-rt}{\lambda_k}}}{2}\sum_{n=1}^{\infty} e^{\frac{tn^2}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\cos\left(\frac{n\pi(x+\xi)}{L}\right)$$

$$=\frac{e^{\frac{-rt}{\lambda_k}}}{4}\left[\vartheta_3\left(\pi\frac{x-\xi}{2L}, e^{\frac{t}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\right)-\vartheta_3\left(\pi\frac{x+\xi}{2L}, e^{\frac{t}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\right)\right]$$

with k = p, p−1, …, 1, −1, … −(p−1). Therefore the Green matrixes are given by:

$$\left[K^{(0)}(x,\xi,t)\right]=\sum_{\substack{k=-(p-1)\\k\neq 0}}^{p}\Gamma_k(x,\xi,t)\mathbf{b}_k\mathbf{b}_k^T \qquad \left[K(x,\xi,t)\right]=\sum_{\substack{k=-(p-1)\\k\neq 0}}^{p}\Gamma_k(x,\xi,t)\frac{\mathbf{b}_k\mathbf{b}_k^T}{\lambda_k}$$

And the solution of (P) results to be:

$$\mathbf{i}(x,t)=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0+\frac{2}{L}\int_0^L d\xi\left[K^{(0)}(x,\xi,t)\right]\mathbf{i}^{(0)}(\xi)+\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\left[K(x,\xi,t-\tau)\right]\mathbf{v}(\xi,\tau)$$

Note that this solution of (P) results to be invariant by addition to the sources $\mathbf{i}^{(0)}(x)$ and $\mathbf{v}(x, t)$ of terms proportional to a $\mathbf{b}_0$. In fact:

$$\left[K^{(0)}(x,\xi)\right]\left\{\mathbf{i}^{(0)}(\xi)+\hat{\imath}(\xi)\mathbf{b}_0\right\}=\left[K^{(0)}(x,\xi)\right]\mathbf{i}^{(0)}(\xi)+\hat{v}(\xi)\left[K^{(0)}(x,\xi)\right]\mathbf{b}_0=\left[K^{(0)}(x,\xi)\right]\mathbf{i}^{(0)}(\xi)$$

$$\left[K(x,\xi,\tau)\right]\left\{\mathbf{v}(\xi,\tau)+\hat{v}(\xi,\tau)\mathbf{b}_0\right\}=\left[K(x,\xi,\tau)\right]\mathbf{v}(\xi,\tau)+\hat{v}(\xi,\tau)\left[K(x,\xi,\tau)\right]\mathbf{b}_0=\left[K(x,\xi,\tau)\right]\mathbf{v}(\xi,\tau)$$

This means that the solution does not change if in the first integral $\mathbf{i}^{(0)}(x)$ is replaced by $\delta\mathbf{i}^{(0)}(x)$ and if the reference node for the voltages is changed.

Note that the calculation of the integration kernels $[K^{(0)}]$ and $[K]$ may cause convergence difficulties, since the function Γ_k approaches the Dirac δ distribution, for t going to zero:

$$\lim_{t\to 0}\Gamma_k(x,\xi;t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)=\frac{L}{2}\delta(x-\xi)$$

Thus, if $\mathbf{i}^{(0)}(x)$ and $\mathbf{v}(x, t)$ are differentiable with respect to x, it's better to utilize an alternative form of the solution, obtained through an integration by parts. Defining the new function $\underline{\Gamma}_k$[(oooo)]:

$$\underline{\Gamma}_k(x,\xi;t)=\int_0^{\xi}\Gamma_k(x,\xi';t)\,d\xi'=2L\sum_{n=1}^{\infty}\frac{e^{\frac{-t}{\lambda_k}\left[r-\frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]}}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\sin^2\left(\frac{n\pi\xi}{2L}\right)$$

[(oooo)] Note that $\underline{\Gamma}_k$ satisfies the same equations of Γ_k: $\gamma_k\lambda_k\frac{\partial\underline{\Gamma}_k}{\partial t}(x,\xi,t)+\gamma_k r\underline{\Gamma}_k(x,\xi,t)+\frac{\partial^2\underline{\Gamma}_k}{\partial x^2}(x,\xi,t)=0$

with k = p, p−1, …, 1, −1, … −(p−1), and the new integration kernels [$\underline{K}^{(0)}$] and [$\underline{K}$]:

$$\left[\underline{K}^{(0)}(x,\xi,t)\right]=\sum_{\substack{k=-(p-1)\\k\neq 0}}^{p}\underline{\Gamma}_k(x,\xi,t)\mathbf{b}_k\mathbf{b}_k^T \qquad \left[\underline{K}(x,\xi,t)\right]=\sum_{\substack{k=-(p-1)\\k\neq 0}}^{p}\underline{\Gamma}_k(x,\xi,t)\frac{\mathbf{b}_k\mathbf{b}_k^T}{\lambda_k}$$

The solution of (P) can be written as:

$$\mathbf{i}(x,t)=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0+\frac{2}{L}\int_0^L d\xi\,\frac{\partial}{\partial\xi}\left[\underline{K}^{(0)}(x,\xi,t)\right]\mathbf{i}^{(0)}(\xi)+\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\,\frac{\partial}{\partial\xi}\left[\underline{K}(x,\xi,t-\tau)\right]\mathbf{v}(\xi,\tau)=$$

$$=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0+\left\{\frac{2}{L}\left[\underline{K}^{(0)}(x,\xi,t)\right]\mathbf{i}^{(0)}(\xi)\right\}_{\xi=0}^{\xi=L}-\frac{2}{L}\int_0^L d\xi\,\left[\underline{K}^{(0)}(x,\xi,t)\right]\frac{\partial\mathbf{i}^{(0)}}{\partial\xi}(\xi)+$$

$$+\frac{2}{L}\left\{\int_0^t d\tau\,\left[\underline{K}(x,\xi,t-\tau)\right]\mathbf{v}(\xi,\tau)\right\}_{\xi=0}^{\xi=L}-\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\,\left[\underline{K}(x,\xi,t-\tau)\right]\frac{\partial\mathbf{v}}{\partial\xi}(\xi,\tau)$$

and finally:

$$\mathbf{i}(x,t)=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0+\frac{2}{L}\left[\underline{K}^{(0)}(x,L,t)\right]\mathbf{i}^{(0)}(L)-\frac{2}{L}\int_0^L d\xi\,\left[\underline{K}^{(0)}(x,\xi,t)\right]\frac{\partial\mathbf{i}^{(0)}}{\partial\xi}(\xi)+$$

$$+\frac{2}{L}\int_0^t d\tau\,\left[\underline{K}(x,L,t-\tau)\right]\mathbf{v}(L,\tau)-\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\,\left[\underline{K}(x,\xi,t-\tau)\right]\frac{\partial\mathbf{v}}{\partial\xi}(\xi,\tau)$$

Note that the calculation of the integration kernels [$K^{(0)}$] and [K] is no more cause of convergence difficulties, since the function $\underline{\Gamma}_k$ approaches the Heaviside step function, for t going to zero:

$$\lim_{t\to 0}\underline{\Gamma}_k(x,\xi;t)=\sum_{n=1}^{\infty}\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\sin^2\left(\frac{n\pi\xi}{2L}\right)=\frac{L}{2}U(\xi-x)=\frac{L}{2}\begin{cases}1, & x<\xi\\ 0, & x>\xi\end{cases}$$

In the particular case in which $\mathbf{i}^{(0)}(x)$ has equal components and $\mathbf{v}(x, t)$ is time-independent, the solution further simplifies, defining:

$$\Gamma_k^*(x,\xi;t)=\int_0^t\Gamma_k(x,\xi;\tau)\,d\tau=\sum_{n=1}^{\infty}\frac{\lambda_k}{\left[r-\frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]}\left\{1-e^{\frac{-t}{\lambda_k}\left[r-\frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]}\right\}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)$$

with k = p, p−1, …, 1, −1, … −(p−1); and the integration kernel [K*]:

$$\left[K^*(x,\xi,t)\right]=\sum_{\substack{k=-(p-1)\\k\neq 0}}^{p}\Gamma_k^*(x,\xi,t)\frac{\mathbf{b}_k\mathbf{b}_k^T}{\lambda_k}$$

the solution of (P) can be written as:

$$\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{N}}\mathbf{b}_0 + \frac{2}{L}\int_0^L d\xi\ [K^*(x,\xi,t)]\,\mathbf{v}(\xi)$$

3. Comparison with the Krempasky-Schmidt solution

The solution found above is now compared with a known one for a two-strand cable {4}. With the conditions:

1- $I(t) = 0$;

2- $r = 0$,

3- even number of periods,

4- $\mathbf{i}^{(0)}(x) = 0$,

5- time-independent excitation spatially bounded excitation to an interval in the center of the cable.

Hypotheses 3 and 5 lead to write:

$$\mathbf{v}(\xi) = \frac{\dot{\Phi}/2}{\delta} U(\xi - \frac{L}{2} - \frac{\delta}{2})U(\frac{L}{2} + \frac{\delta}{2} - \xi)\begin{Bmatrix}+1\\-1\end{Bmatrix}$$

where U is the Heaviside step function. For two strand, N = 2, p = 1; thus, given

$$[M] = \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{11}\end{bmatrix} \qquad [G] = \begin{bmatrix} -g_{12} & +g_{12} \\ +g_{12} & -g_{12}\end{bmatrix}$$

1 - the eigenvalues of [M] are positive and given by:

$$\lambda_0 = m_{11} + m_{12}\ ; \qquad \lambda_1 = m_{11} - m_{12}$$

2 - the eigenvalues of [G] are negative (except for one that is null) e given by:

$$\gamma_0 = 0\ ; \qquad \gamma_1 = -2g_{12}$$

3 - the orthonormal spectral basis $\mathbf{b}_k$, with k = 1, 0, it's the same for [M] and [G]:

$$\mathbf{b}_0^T = \left\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\} \qquad \mathbf{b}_1^T = \left\{\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right\}$$

To reproduce the nomenclature of {4} the following variables are defined:

$L_1 = 2\ (m_{11} - m_{12})$, $G_1 = g_{12}$, $\alpha = \pi\ w/(2w + \delta)$, $L = 2w + \delta$, $\tau = \frac{4}{\pi^2} L_1 G_1 \left(\frac{2w+\delta}{2}\right)^2$

(Note that $\pi / L = \alpha / w$, $L / w = \pi / \alpha$ and $(\pi\delta / 2L) + \alpha = \pi / 2$)

Thus we get: $\mathbf{v}(\xi) = \frac{\dot{\Phi}}{2\delta} U(\xi - \frac{L}{2} - \frac{\delta}{2})U(\frac{L}{2} + \frac{\delta}{2} - \xi)\mathbf{b}_1\sqrt{2}$. When the simplified solution is utilized, we have:

$$\Gamma_1^*(x,\xi;t)=\sum_{n=1}^{\infty}\frac{\lambda_1}{\left[-\frac{1}{\gamma_1}\left(\frac{n\pi}{L}\right)^2\right]}\left\{1-e^{\frac{-t}{\lambda_1}\left[-\frac{1}{\gamma_1}\left(\frac{n\pi}{L}\right)^2\right]}\right\}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)=$$

$$=\frac{2g_{12}\lambda_1 L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)$$

(Note that

$$\frac{1}{\lambda_1}\left[-\frac{1}{\gamma_1}\left(\frac{\pi}{L}\right)^2\right]=\frac{1}{m_{11}-m_{12}}\left[\frac{1}{2g_{12}}\left(\frac{\pi}{L}\right)^2\right]=\frac{1}{L_1G_1}\frac{\pi^2}{(2w+\delta)^2}=\frac{\pi^2}{4}\frac{1}{L_1G_1}\left(\frac{2}{2w+\delta}\right)^2=\frac{1}{\tau})$$

Therefore the kernel becomes:

$$[K^*(x,\xi,t)]=\Gamma_1^*(x,\xi,t)\frac{\boldsymbol{b}_1\boldsymbol{b}_1^T}{\lambda_1}=\frac{\Gamma_1^*(x,\xi,t)}{\lambda_1}\begin{Bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{Bmatrix}\left\{\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\}=$$

$$=\frac{\Gamma_1^*(x,\xi,t)}{\lambda_1}\begin{bmatrix}+1/2 & -1/2\\-1/2 & +1/2\end{bmatrix}=\frac{\Gamma_1^*(x,\xi,t)}{2\lambda_1}\begin{bmatrix}+1 & -1\\-1 & +1\end{bmatrix}=$$

$$=\frac{g_{12}L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)\begin{bmatrix}+1 & -1\\-1 & +1\end{bmatrix}$$

and the solution can be written as:

$$\mathbf{i}(x,t)=\frac{2}{L}\int_0^L d\xi\ [K^*(x,\xi,t)]\,\mathbf{v}(\xi)=$$

$$=\frac{2g_{12}L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\pi x}{L}\right)\int_0^L d\xi\ \sin\left(\frac{n\pi\xi}{L}\right)\begin{bmatrix}+1 & -1\\-1 & +1\end{bmatrix}\mathbf{v}(\xi)=$$

$$=\frac{\dot{\Phi}g_{12}L}{\pi^2\delta}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\pi x}{L}\right)\int_{(L-\delta)/2}^{(L+\delta)/2} d\xi\ \sin\left(\frac{n\pi\xi}{L}\right)\begin{bmatrix}+1 & -1\\-1 & +1\end{bmatrix}\begin{Bmatrix}+1\\-1\end{Bmatrix}=$$

$$\frac{\dot{\Phi}g_{12}L}{\pi^2\delta}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\pi x}{L}\right)\int_{(L-\delta)/2}^{(L+\delta)/2} d\xi\ \sin\left(\frac{n\pi\xi}{L}\right)\begin{Bmatrix}+2\\-2\end{Bmatrix}$$

Since $i_2 = -i_1$, only $i_1(x, t)$ is considered:

$$i_1(x,t)=\frac{2\dot{\Phi}G_1L}{\pi^2\delta}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\pi x}{L}\right)\int_{(L-\delta)/2}^{(L+\delta)/2} d\xi\ \sin\left(\frac{n\pi\xi}{L}\right)=$$

$$=\frac{4I_mL}{\pi^2 w\delta}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\alpha x}{w}\right)\int_{(L-\delta)/2}^{(L+\delta)/2} d\xi\ \sin\left(\frac{n\pi\xi}{L}\right)=$$

$$=\frac{4}{\pi\alpha}I_m\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\alpha x}{w}\right)\frac{1}{\delta}\int_{(L-\delta)/2}^{(L+\delta)/2} d\xi\ \sin\left(\frac{n\pi\xi}{L}\right)$$

where $I_m = \dot{\Phi} G_1 w / 2$. The last integral can be made explicit, as follows:

$$\frac{1}{\delta}\int_{(L-\delta)/2}^{(L+\delta)/2} d\xi \,\sin\left(\frac{n\pi\xi}{L}\right) = \frac{L}{n\pi\delta}\left[\cos\left(n\pi\frac{L-\delta}{2L}\right) - \cos\left(n\pi\frac{L+\delta}{2L}\right)\right] =$$

$$= \frac{2L}{n\pi\delta}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi\delta}{2L}\right)$$

Note that, since sin ($n\pi/2$) = 0 for even n, it's possible to restrict la summation to the odd n only. Moreover, note that

$$\sin\left(\frac{n\pi}{2}\right) = \sin\left(\frac{n\pi\delta}{2L} + n\alpha\right) = \sin\left(\frac{n\pi\delta}{2L}\right)\cos(n\alpha) + \cos\left(\frac{n\pi\delta}{2L}\right)\sin(n\alpha)$$

Thus, assuming $\delta \ll L$:

$$\frac{1}{\delta}\int_{(L-\delta)/2}^{(L+\delta)/2} d\xi \,\sin\left(\frac{n\pi\xi}{L}\right) = \frac{\sin\left(\frac{n\pi\delta}{2L}\right)}{\left(\frac{n\pi\delta}{2L}\right)}\left[\sin\left(\frac{n\pi\delta}{2L}\right)\cos(n\alpha) + \cos\left(\frac{n\pi\delta}{2L}\right)\sin(n\alpha)\right] \cong \sin(n\alpha)$$

Note that this corresponds to a first order approximation of the original integral:

$$\frac{1}{\delta}\int_{(L-\delta)/2}^{(L+\delta)/2} d\xi \,\sin\left(\frac{n\pi\xi}{L}\right) = \frac{1}{\delta}\int_{w}^{w+\delta} d\xi \,\sin\left(\frac{n\pi\xi}{2w+\delta}\right) \cong \frac{1}{\delta}\sin\left(\frac{n\pi w}{2w+\delta}\right)\delta = \sin(n\alpha)$$

and finally:

$$i_1(x,t) = \frac{4}{\pi\alpha} I_m \sum_{\substack{n=1 \\ \text{odd } n}}^{\infty} \frac{1}{n^2}\left\{1 - e^{-tn^2/\tau}\right\}\sin\left(\frac{n\alpha x}{w}\right)\sin(n\alpha)$$

If the external field ramp is stopped at time t_1 and the field is kept constant, the external voltage drops to zero. In this case the induced currents start decaying. In order to prove this, we start noting that, since the external voltage drop to zero for $t > t_1$, the external voltage is time dependent and it can found multiplying $\mathbf{v}(\xi)$ by $U(t_1 - t)$. Thus, the solution can be written as follows:

$$\mathbf{i}(x,t) = \frac{2}{L}\int_0^L d\xi \int_0^t d\tau \,\mathbf{K}(x,\xi,t-\tau)\,\mathbf{v}(\xi) U(t_1 - \tau)$$

Therefore for $t > t_1$ this equation gives

$$\mathbf{i}(x,t) = \frac{2}{L}\int_0^L d\xi \int_0^{t_1} d\tau \,\mathbf{K}(x,\xi,t-\tau)\,\mathbf{v}(\xi) =$$

$$= \frac{2}{L}\int_0^L d\xi \left(\int_{t-t_1}^{t} dt' \,\mathbf{K}(x,\xi,t')\right)\mathbf{v}(\xi) =$$

$$= \frac{2}{L}\int_0^L d\xi \,\mathbf{K}^*(x,\xi,t)\,\mathbf{v}(\xi) - \frac{2}{L}\int_0^L d\xi \,\mathbf{K}^*(x,\xi,t-t_1)\,\mathbf{v}(\xi)$$

These last two integrals are the same previously solved. Thus, we can evaluate directly them as follows:

$$\mathbf{i}(x,t)=\frac{4}{\pi\alpha}I_m\sum_{\substack{n=1\\ \text{odd } n}}^{\infty}\frac{1}{n^2}\left\{1-e^{-t/\tau_n}\right\}\sin\left(\frac{n\alpha x}{w}\right)\sin(n\alpha)\mathbf{b}_1+$$

$$-\frac{4}{\pi\alpha}I_m\sum_{\substack{n=1\\ \text{odd } n}}^{\infty}\frac{1}{n^2}\left\{1-e^{-(t-t_1)/\tau_n}\right\}\sin\left(\frac{n\alpha x}{w}\right)\sin(n\alpha)\mathbf{b}_1=$$

$$=\frac{4}{\pi\alpha}I_m\sum_{\substack{n=1\\ \text{odd } n}}^{\infty}\frac{1}{n^2}\left\{e^{-(t-t_1)/\tau_n}-e^{-t/\tau_n}\right\}\sin\left(\frac{n\alpha x}{w}\right)\sin(n\alpha)\mathbf{b}_1$$

Therefore, factorizing out the exponential, it can be seen that each component under the sum decays with its corresponding time constant τ_n. Thus, the strand currents are given by:

$$\mathbf{i}(x,t)=\left[\frac{4}{\pi\alpha}I_m\sum_{\substack{n=1\\ n\ odd}}^{\infty}\frac{1}{n^2}\left(1-e^{-t_1/\tau_n}\right)\sin\left(\frac{n\alpha\, x}{w}\right)\sin\,(n\alpha)\,e^{-(t-t_1)/\tau_n}\right]\mathbf{b}_1$$

4. Convergence Analysis

The calculation of the function Γ_k may cause convergence difficulties, since the summation terms are oscillating.

$$\Gamma_k(x,\xi;t)=\sum_{n=1}^{\infty}e^{\frac{-t}{\lambda_k}\left[r-\frac{1}{\gamma_k}\left(\frac{n\pi}{L}\right)^2\right]}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)=$$

$$=\frac{e^{\frac{-rt}{\lambda_k}}}{2}\sum_{n=1}^{\infty}e^{\frac{tn^2}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\cos\left(\frac{n\pi(x-\xi)}{L}\right)-\frac{e^{\frac{-rt}{\lambda_k}}}{2}\sum_{n=1}^{\infty}e^{\frac{tn^2}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\cos\left(\frac{n\pi(x+\xi)}{L}\right)$$

$$=\frac{e^{\frac{-rt}{\lambda_k}}}{4}\left[\vartheta_3\left(\pi\frac{x-\xi}{2L},e^{\frac{t}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\right)-\vartheta_3\left(\pi\frac{x+\xi}{2L},e^{\frac{t}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\right)\right]$$

It can be demonstrated {5} that the elliptic theta function ϑ_3 admit the following representation (where the summation terms are non-oscillating):

$$\vartheta_3\left(u,e^{-s}\right)=\sum_{n=-\infty}^{\infty}e^{-n^2 s}\cos 2nu=\sqrt{\frac{\pi}{s}}\sum_{n=-\infty}^{\infty}e^{-\frac{(u-n\pi)^2}{s}}$$

Thus the function Γ_k admits the following alternative representation:

$$\Gamma_k(x,\xi;t)=\frac{e^{\frac{-rt}{\lambda_k}}}{4}\left[\vartheta_3\left(\pi\frac{x-\xi}{2L},e^{\frac{t}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\right)-\vartheta_3\left(\pi\frac{x+\xi}{2L},e^{\frac{t}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\right)\right]=$$

$$=L\frac{e^{\frac{-rt}{\lambda_k}}}{4}\sqrt{-\frac{\lambda_k\gamma_k}{\pi t}}\sum_{n=-\infty}^{\infty}\left[e^{L^2\frac{\lambda_k\gamma_k}{\pi^2 t}\left(\pi\frac{x-\xi}{2L}-n\pi\right)^2}-e^{L^2\frac{\lambda_k\gamma_k}{\pi^2 t}\left(\pi\frac{x+\xi}{2L}-n\pi\right)^2}\right]=$$

$$=L\frac{e^{\frac{-rt}{\lambda_k}}}{4}\sqrt{-\frac{\lambda_k\gamma_k}{\pi t}}\sum_{n=-\infty}^{\infty}\left[e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2}-e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2}\right]=$$

$$=\frac{L}{4}\sqrt{-\frac{\lambda_k\gamma_k}{\pi t}}\sum_{n=-\infty}^{\infty}\left[e^{\frac{-rt}{\lambda_k}+\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2}-e^{\frac{-rt}{\lambda_k}+\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2}\right]$$

Moreover, for what concerns Γ_k*, we get, after defining $\theta=\sqrt{\tau}\quad\Rightarrow\quad 2d\theta=\frac{d\tau}{\sqrt{\tau}}$:

$$\Gamma_k^*(x,\xi;t)=\int_0^t\Gamma_k(x,\xi;\tau)\,d\tau=\int_0^t\frac{L}{4}\sqrt{-\frac{\lambda_k\gamma_k}{\pi\tau}}\sum_{n=-\infty}^{\infty}\left[e^{\frac{-r\tau}{\lambda_k}+\frac{\lambda_k\gamma_k}{\tau}\left(\frac{x-\xi}{2}-nL\right)^2}-e^{\frac{-r\tau}{\lambda_k}+\frac{\lambda_k\gamma_k}{\tau}\left(\frac{x+\xi}{2}-nL\right)^2}\right]d\tau=$$

$$=\frac{L}{2}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}}\int_0^{\sqrt{t}}d\theta\sum_{n=-\infty}^{\infty}\left[e^{\frac{-r\theta^2}{\lambda_k}+\frac{\lambda_k\gamma_k}{\theta^2}\left(\frac{x-\xi}{2}-nL\right)^2}-e^{\frac{-r\theta^2}{\lambda_k}+\frac{\lambda_k\gamma_k}{\theta^2}\left(\frac{x+\xi}{2}-nL\right)^2}\right]=$$

$$=\sum_{n=-\infty}^{\infty}\left[\frac{L}{2}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}}\int_0^{\sqrt{t}}e^{\frac{-r\theta^2}{\lambda_k}+\frac{\lambda_k\gamma_k}{\theta^2}\left(\frac{x-\xi}{2}-nL\right)^2}d\theta-\frac{L}{2}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}}\int_0^{\sqrt{t}}e^{\frac{-r\theta^2}{\lambda_k}+\frac{\lambda_k\gamma_k}{\theta^2}\left(\frac{x+\xi}{2}-nL\right)^2}d\theta\right]$$

Consider now the summation term:

$$\frac{L}{2}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}}\int_0^{\sqrt{t}}e^{\frac{-r\theta^2}{\lambda_k}+\frac{\lambda_k\gamma_k}{\theta^2}Q^2}d\theta=\frac{L}{2}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}}e^{-2|Q|\sqrt{-r\gamma_k}}\int_0^{\sqrt{t}}e^{-\left[\theta\sqrt{\frac{r}{\lambda_k}}-\frac{\sqrt{-\lambda_k\gamma_k}}{\theta}|Q|\right]^2}d\theta$$

Defining $y=\theta\sqrt{\frac{r}{\lambda_k}}-\sqrt{-\lambda_k\gamma_k}\frac{|Q|}{\theta}\quad\Rightarrow\quad\theta=\frac{y}{2}\sqrt{\frac{\lambda_k}{r}}+\sqrt{\frac{y^2}{4}\frac{\lambda_k}{r}+\lambda_k|Q|\sqrt{\frac{-\gamma_k}{r}}}$ we have that:

$$d\theta=\frac{1}{2}\sqrt{\frac{\lambda_k}{r}}\left(1+\frac{y}{\sqrt{y^2+4|Q|\sqrt{-r\gamma_k}}}\right)dy$$

Therefore:

$$\frac{L}{2}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}}\int_0^{\sqrt{t}}e^{\frac{-r\theta^2}{\lambda_k}+\frac{\lambda_k\gamma_k}{\theta^2}Q^2}d\theta=\frac{L\lambda_k}{4}\sqrt{-\frac{\gamma_k}{\pi r}}e^{-2|Q|\sqrt{-r\gamma_k}}\int_{-\infty}^{\sqrt{\frac{rt}{\lambda_k}}-\sqrt{\frac{-\lambda_k\gamma_k}{t}}|Q|}e^{-y^2}\left(1+\frac{y}{\sqrt{y^2+4|Q|\sqrt{-r\gamma_k}}}\right)dy$$

Defining the functions $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ and erfc (x) = 1 − erf (x), we have:

$$\frac{L}{2}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}} \int_0^{\sqrt{t}} e^{\frac{-r\theta^2}{\lambda_k}+\frac{\lambda_k\gamma_k}{\theta^2}Q^2} d\theta = \frac{L\lambda_k}{4}\sqrt{-\frac{\gamma_k}{\pi r}} e^{-2|Q|\sqrt{-r\gamma_k}} \Bigg[-\frac{\sqrt{\pi}}{2}\mathrm{erf}(-\infty) + \frac{\sqrt{\pi}}{2}\mathrm{erf}\left(\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}|Q|\right) +$$
$$- \int_0^{-\infty} \frac{y e^{-y^2}}{\sqrt{y^2 + 4|Q|\sqrt{-r\gamma_k}}} dy + \int_0^{\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}|Q|} \frac{y e^{-y^2}}{\sqrt{y^2 + 4|Q|\sqrt{-r\gamma_k}}} dy \Bigg]$$

Defining $w^2 = y^2 + 4|Q|\sqrt{-r\gamma_k} \quad \Rightarrow \quad wdw = ydy$ we have:

$$\int_0^X \frac{y e^{-y^2}}{\sqrt{y^2 + 4|Q|\sqrt{-r\gamma_k}}} dy = e^{4|Q|\sqrt{-r\gamma_k}} \int_{\sqrt{4|Q|\sqrt{-r\gamma_k}}}^{\sqrt{X^2+4|Q|\sqrt{-r\gamma_k}}} e^{-w^2} dw =$$
$$= \frac{\sqrt{\pi}}{2}\left[\mathrm{erf}\left(\sqrt{X^2 + 4|Q|\sqrt{-r\gamma_k}}\right) - \mathrm{erf}\left(\sqrt{4|Q|\sqrt{-r\gamma_k}}\right)\right]$$

Thus:

$$\frac{L}{2}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}} \int_0^{\sqrt{t}} e^{\frac{-r\theta^2}{\lambda_k}+\frac{\lambda_k\gamma_k}{\theta^2}Q^2} d\theta = \frac{L\lambda_k}{8}\sqrt{-\frac{\gamma_k}{r}} e^{-2|Q|\sqrt{-r\gamma_k}} \Bigg[1 + \mathrm{erf}\left(\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}|Q|\right) +$$
$$- e^{4|Q|\sqrt{-r\gamma_k}} + e^{4|Q|\sqrt{-r\gamma_k}} \mathrm{erf}\left(\sqrt{\frac{rt}{\lambda_k}} + \sqrt{\frac{-\lambda_k\gamma_k}{t}}|Q|\right)\Bigg] =$$
$$= \frac{L\lambda_k}{8}\sqrt{-\frac{\gamma_k}{r}}\left[2 e^{-2|Q|\sqrt{-r\gamma_k}} - e^{-2|Q|\sqrt{-r\gamma_k}}\mathrm{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}|Q|\right) - e^{2|Q|\sqrt{-r\gamma_k}}\mathrm{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} + \sqrt{\frac{-\lambda_k\gamma_k}{t}}|Q|\right)\right]$$

and finally:

$$\Gamma_k^*(x,\xi;t) = \frac{L\lambda_k}{8}\sqrt{-\frac{\gamma_k}{r}} \sum_{n=-\infty}^{\infty} \Bigg[2 e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}} - 2 e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}} +$$
$$- e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}\mathrm{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2} - nL\right|\right) - e^{|x-\xi-2nL|\sqrt{-r\gamma_k}}\mathrm{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} + \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2} - nL\right|\right) +$$
$$+ e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\mathrm{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2} - nL\right|\right) + e^{|x+\xi-2nL|\sqrt{-r\gamma_k}}\mathrm{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} + \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2} - nL\right|\right)\Bigg]$$

Note that the first two terms are time-independent and, therefore, they represent the regime solution. Thus:

$$\Gamma_k^*(x,\xi;t) = \Gamma_k^*(x,\xi;\infty) + \frac{L\lambda_k}{8}\sqrt{-\frac{\gamma_k}{r}}\sum_{n=-\infty}^{\infty}$$
$$\left[- e^{-\left|x-\xi-2nL\right|\sqrt{-r\gamma_k}} \operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right) - e^{\left|x-\xi-2nL\right|\sqrt{-r\gamma_k}} \operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} + \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right) + \right.$$
$$\left. + e^{-\left|x+\xi-2nL\right|\sqrt{-r\gamma_k}} \operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right) + e^{\left|x+\xi-2nL\right|\sqrt{-r\gamma_k}} \operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} + \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right)\right]$$

Note that the function Γ_k^* has two (unessential) singularity points for $t \to 0$ and for $t \to +\infty$; in fact $\operatorname{erfc}(-\infty) \to 2$ and $\operatorname{erfc}(+\infty) \to 0$.

5. Regime Solution

Note that since $\lim_{t\to 0}\Gamma_k^*(x,\xi;t) = 0$ it's possible to evaluate easily the regime value of Γ_k^*. In fact:

$$\Gamma_k^*(x,\xi;\infty) = \int_0^\infty \frac{L}{4}\sqrt{-\frac{\lambda_k\gamma_k}{\pi\tau}}\sum_{n=-\infty}^{\infty}\left[e^{\frac{-r\tau}{\lambda_k}+\frac{\lambda_k\gamma_k}{\tau}\left(\frac{x-\xi}{2}-nL\right)^2} - e^{\frac{-r\tau}{\lambda_k}+\frac{\lambda_k\gamma_k}{\tau}\left(\frac{x+\xi}{2}-nL\right)^2}\right]d\tau =$$

$$= \frac{L}{4}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}}\sum_{n=-\infty}^{\infty}\left[\int_0^\infty \frac{e^{\frac{-r\tau}{\lambda_k}+\frac{\lambda_k\gamma_k}{\tau}\left(\frac{x-\xi}{2}-nL\right)^2}}{\sqrt{\tau}}d\tau - \int_0^\infty \frac{e^{\frac{-r\tau}{\lambda_k}+\frac{\lambda_k\gamma_k}{\tau}\left(\frac{x+\xi}{2}-nL\right)^2}}{\sqrt{\tau}}d\tau\right] =$$

$$= \frac{L}{4}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}}\sum_{n=-\infty}^{\infty}\left[\mathcal{L}_{s=\frac{-r}{\lambda_k}}\left\{\frac{e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2}}{\sqrt{t}}\right\} - \mathcal{L}_{s=\frac{-r}{\lambda_k}}\left\{\frac{e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2}}{\sqrt{t}}\right\}\right]$$

Recalling that $\mathcal{L}_s\left\{\frac{e^{-a/4t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\,e^{-\sqrt{as}}$ {6}, it follows:

$$\Gamma_k^*(x,\xi;\infty) = \frac{L}{4}\sqrt{-\frac{\lambda_k\gamma_k}{\pi}}\sum_{n=-\infty}^{\infty}\left[\mathcal{L}_{s=\frac{-r}{\lambda_k}}\left\{\frac{e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2}}{\sqrt{t}}\right\} - \mathcal{L}_{s=\frac{-r}{\lambda_k}}\left\{\frac{e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2}}{\sqrt{t}}\right\}\right] =$$

$$= \frac{L\lambda_k}{4}\sqrt{-\frac{\gamma_k}{r}}\sum_{n=-\infty}^{\infty}\left[e^{-2\sqrt{-r\gamma_k}\left|\frac{x-\xi}{2}-nL\right|} - e^{-2\sqrt{-r\gamma_k}\left|\frac{x+\xi}{2}-nL\right|}\right]$$

Since $0 < x < L$ and $0 < \xi < L$, it's simple to show that, separating the summation on the positive and negative indexes, we have

$$\Gamma_k^*(x,\xi;\infty)=\frac{L\lambda_k}{4}\sqrt{-\frac{\gamma_k}{r}}\left\{\sum_{n=-\infty}^{-1}e^{-2\sqrt{-r\gamma_k}\left(\frac{x-\xi}{2}-nL\right)}-\sum_{n=-\infty}^{-1}e^{-2\sqrt{-r\gamma_k}\left(\frac{x+\xi}{2}-nL\right)}\right.$$

$$\left.+e^{-2\sqrt{-r\gamma_k}\left|\frac{x-\xi}{2}\right|}-e^{-2\sqrt{-r\gamma_k}\left(\frac{x+\xi}{2}\right)}+\sum_{n=1}^{\infty}e^{2\sqrt{-r\gamma_k}\left(\frac{x-\xi}{2}-nL\right)}-\sum_{n=1}^{\infty}e^{2\sqrt{-r\gamma_k}\left(\frac{x+\xi}{2}-nL\right)}\right\}=$$

$$=\frac{L\lambda_k}{4}\sqrt{-\frac{\gamma_k}{r}}\left\{e^{-(x-\xi)\sqrt{-r\gamma_k}}\sum_{n=1}^{\infty}\left(e^{-2L\sqrt{-r\gamma_k}}\right)^n-e^{-(x+\xi)\sqrt{-r\gamma_k}}\sum_{n=1}^{\infty}\left(e^{-2L\sqrt{-r\gamma_k}}\right)^n\right.$$

$$\left.+e^{-|x-\xi|\sqrt{-r\gamma_k}}-e^{-(x+\xi)\sqrt{-r\gamma_k}}+e^{-(x-\xi)\sqrt{-r\gamma_k}}\sum_{n=1}^{\infty}\left(e^{-2L\sqrt{-r\gamma_k}}\right)^n-e^{-(x+\xi)\sqrt{-r\gamma_k}}\sum_{n=1}^{\infty}\left(e^{-2L\sqrt{-r\gamma_k}}\right)^n\right\}=$$

$$=\frac{L\lambda_k}{2}\sqrt{-\frac{\gamma_k}{r}}\left\{\frac{\cosh(x-\xi)\sqrt{-r\gamma_k}-\cosh(x+\xi)\sqrt{-r\gamma_k}}{e^{2L\sqrt{-r\gamma_k}}-1}+\frac{e^{-|x-\xi|\sqrt{-r\gamma_k}}-e^{-(x+\xi)\sqrt{-r\gamma_k}}}{2}\right\}$$

where it has been used the fact that $\sum_{n=1}^{\infty}q^n=\frac{1}{\frac{1}{q}-1}$. Finally:

$$\boxed{\Gamma_k^*(x,\xi;\infty)=L\lambda_k\sqrt{-\frac{\gamma_k}{r}}\left\{\frac{e^{-|x-\xi|\sqrt{-r\gamma_k}}-e^{-(x+\xi)\sqrt{-r\gamma_k}}}{4}-\frac{\sinh\left(x\sqrt{-r\gamma_k}\right)\sinh\left(\xi\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}\right\}}$$

6. Non-linear Case

If the resistance of the strand is a known function of the current, i.e.

$$r_j=\hat{r}\left(i_j(x,t)\right),\ \text{with } j = 1, \ldots, N$$

the problem (P) can still reduced to a diagonal form. Assigning

$$[M]=\sum_{k=-(p-1)}^{p}\lambda_k\mathbf{b}_k\mathbf{b}_k^T\qquad[G]=\sum_{k=-(p-1)}^{p}\gamma_k\mathbf{b}_k\mathbf{b}_k^T\qquad[R]=\sum_{j=1}^{N}r_j\,\mathbf{e}_j\mathbf{e}_j^T$$

where $\mathbf{e}_j$ denote the unit vectors of the Cartesian basis, we have (with $k \neq 0$):

$$0=[G][M]\frac{\partial\mathbf{i}}{\partial t}(x,t)+[G][R]\mathbf{i}(x,t)+\frac{\partial^2\mathbf{i}}{\partial x^2}(x,t)-[G]\mathbf{v}(x,t)=$$

$$=\mathbf{b}_k\left\{\gamma_k\lambda_k\frac{\partial}{\partial t}\left[\mathbf{b}_k^T\mathbf{i}(x,t)\right]+\gamma_k\sum_{h=-(p-1)}^{p}\sum_{j=1}^{N}r_j\left[\mathbf{b}_k^T\mathbf{e}_j\right]\left[\mathbf{e}_j^T\mathbf{b}_h\right]\left[\mathbf{b}_h^T\mathbf{i}(x,t)\right]+\frac{\partial^2}{\partial x^2}\left[\mathbf{b}_k^T\mathbf{i}(x,t)\right]-\gamma_k\left[\mathbf{b}_k^T\mathbf{v}(x,t)\right]\right\}$$

and the following system of N–1 equations (k = –(p–1), …,–1, 1, …, p) is obtained:

$$(P_k)\begin{cases}\lambda_k \dfrac{\partial}{\partial t}\left[\mathbf{b}_k^T\mathbf{i}(x,t)\right]+\dfrac{1}{\gamma_k}\dfrac{\partial^2}{\partial x^2}\left[\mathbf{b}_k^T\mathbf{i}(x,t)\right]=\left[\mathbf{b}_k^T\mathbf{v}(x,t)\right]-\sum_{h=-(p-1)}^{p}\left\{\sum_{j=1}^{N} r_j\left[\mathbf{b}_k^T\mathbf{e}_j\right]\left[\mathbf{e}_j^T\mathbf{b}_h\right]\right\}\left[\mathbf{b}_h^T\mathbf{i}(x,t)\right]\\ \mathbf{b}_k^T\mathbf{i}(x=0,t)=\mathbf{b}_k^T\mathbf{i}(x=L,t)=0\\ \mathbf{b}_k^T\mathbf{i}(x,t=0)=\mathbf{b}_k^T\mathbf{i}^{(0)}(x)\end{cases}$$

Nevertheless, the equations are still coupled since $r_j=\hat{r}\left(i_j(x,t)\right)=\hat{r}\left(\sum_{h=-(p-1)}^{p}\left[\mathbf{e}_j^T\mathbf{b}_h\right]\left[\mathbf{b}_h^T\mathbf{i}(x,t)\right]\right)$, with j = 1, …, N. Anyway, note that for k = 0, we get:

$$\begin{cases}\dfrac{\partial^2}{\partial x^2}\left[\mathbf{b}_0^T\mathbf{i}(x,t)\right]=0\\ \mathbf{b}_0^T\mathbf{i}(x=0,t)=\mathbf{b}_0^T\mathbf{i}(x=L,t)=\dfrac{I(t)}{\sqrt{N}}\\ \mathbf{b}_0^T\mathbf{i}(x,t=0)=\mathbf{b}_0^T\mathbf{i}^{(0)}(x)=\dfrac{I(t=0)}{\sqrt{N}}\end{cases}$$

with the usual solution: $\mathbf{b}_0^T\mathbf{i}(x,t)=\dfrac{I(t)}{\sqrt{N}}$. Thus, the solution of problem (P) results to be:

$$\mathbf{i}(x,t)=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0+\sum_{\substack{k=-(p-1)\\ k\neq 0}}^{p}\mathbf{b}_k\left(\mathbf{b}_k^T\mathbf{i}(x,t)\right)$$

Finally, in the particular case for which

$$r_j=\hat{r}(I(t)),\ \text{with } j = 1, \dots, N$$

it's still possible to solve exactly the problem (P). In fact, with the same symbols of §2:

$$\begin{cases}\lambda_k\dfrac{d\eta_{n,k}}{dt}+\left[\hat{r}(I(t))-\dfrac{1}{\gamma_k}\left(\dfrac{n\pi}{L}\right)^2\right]\eta_{n,k}=\nu_{n,k}(t)\\ \eta_{n,k}(t=0)=\mathbf{b}_k^T\mathbf{i}_n^{(0)}\end{cases}\Rightarrow$$

$$\Rightarrow \eta_{n,k}(t)=\left(\mathbf{b}_k^T\mathbf{i}_n^{(0)}\right)e^{\frac{-1}{\lambda_k}\int_0^t\hat{r}(I(t'))dt'+\frac{t}{\lambda_k\gamma_k}\left(\frac{n\pi}{L}\right)^2}+\frac{1}{\lambda_k}\int_0^t\nu_{n,k}(\tau)\,e^{\frac{-1}{\lambda_k}\int_\tau^t\hat{r}(I(t'))dt'+\frac{(t-\tau)}{\lambda_k\gamma_k}\left(\frac{n\pi}{L}\right)^2}d\tau$$

Thus the solution, formally, is unchanged, provided that the function Γ_k is defined as:

$$\Gamma_k(x,\xi;t,\tau)=\sum_{n=1}^{\infty}e^{\frac{-1}{\lambda_k}\int_\tau^{\tau+t}\hat{r}(I(t'))dt'+\frac{t}{\lambda_k\gamma_k}\left(\frac{n\pi}{L}\right)^2}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)=$$

$$=\frac{e^{\frac{-1}{\lambda_k}\int_\tau^{\tau+t}\hat{r}(I(t'))dt'}}{4}\left[\vartheta_3\left(\pi\frac{x-\xi}{2L},e^{\frac{t}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\right)-\vartheta_3\left(\pi\frac{x+\xi}{2L},e^{\frac{t}{\lambda_k\gamma_k}\left(\frac{\pi}{L}\right)^2}\right)\right]$$

Therefore the Green matrixes are expresses by:

$$\left[K^{(0)}(x,\xi,t,0)\right]=\sum_{\substack{k=-(p-1)\\k\neq 0}}^{p}\Gamma_k(x,\xi,t,0)\mathbf{b}_k\mathbf{b}_k^T \qquad \left[K(x,\xi,t,\tau)\right]=\sum_{\substack{k=-(p-1)\\k\neq 0}}^{p}\Gamma_k(x,\xi,t,\tau)\frac{\mathbf{b}_k\mathbf{b}_k^T}{\lambda_k}$$

and the solution of (P) can be written as:

$$\mathbf{i}(x,t)=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0+\frac{2}{L}\int_0^L d\xi\ \left[K^{(0)}(x,\xi,t,0)\right]\mathbf{i}^{(0)}(\xi)+\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\ \left[K(x,\xi,t-\tau,\tau)\right]\mathbf{v}(\xi,\tau)$$

7. Flux Ramp

In the particular case in which $\mathbf{i}^{(0)}(x)$ has equal components and $\mathbf{v}(x, t)$ is given by:

$$\mathbf{v}(x,t)=\begin{cases}\mathbf{v}^*(x), & \text{if}\quad t<t_0\\ 0, & \text{if}\quad t>t_0\end{cases}$$

the solution of (P) can be written as:

$$\mathbf{i}(x,t)=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0+\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\ \left[K(x,\xi,t-\tau)\right]\mathbf{v}(\xi,\tau)$$

that can be specialized in two cases:

I) $t < t_0$

$$\int_0^L d\xi\int_0^t d\tau\ \left[K(x,\xi,t-\tau)\right]\mathbf{v}(\xi,\tau)=\int_0^L d\xi\left(\int_0^t d\tau\ \left[K(x,\xi,t-\tau)\right]\right)\mathbf{v}^*(\xi)=\int_0^L d\xi\left[K^*(x,\xi,t)\right]\mathbf{v}^*(\xi)$$

II) $t > t_0$

$$\int_0^L d\xi\int_0^t d\tau\ \left[K(x,\xi,t-\tau)\right]\mathbf{v}(\xi,\tau)=\int_0^L d\xi\int_0^{t_0} d\tau\ \left[K(x,\xi,t-\tau)\right]\mathbf{v}(\xi,\tau)=\int_0^L d\xi\left(\int_{t-t_0}^t dt'\ \left[K(x,\xi,t')\right]\right)\mathbf{v}^*(\xi)=$$

$$=\int_0^L d\xi\left[K^*(x,\xi,t)\right]\mathbf{v}^*(\xi)-\int_0^L d\xi\left[K^*(x,\xi,t-t_0)\right]\mathbf{v}^*(\xi)$$

thus, after same manipulation, we obtain:

I) $t < t_0$

$$\mathbf{i}(x,t)=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0+\frac{2}{L}\int_0^L d\xi\left[K^*(x,\xi,t)\right]\mathbf{v}^*(\xi)$$

II) $t > t_0$

$$\mathbf{i}(x,t)=\frac{I(t)}{\sqrt{N}}\mathbf{b}_0+\frac{2}{L}\int_0^L d\xi\left[K^*(x,\xi,t)\right]\mathbf{v}^*(\xi)-\frac{2}{L}\int_0^L d\xi\left[K^*(x,\xi,t-t_0)\right]\mathbf{v}^*(\xi)$$

In particular, the last integral can be written as:

$$\frac{2}{L}\int_0^L d\xi [K^*(x,\xi,t-t_0)]\,\mathbf{v}^*(\xi) = -\frac{I(t-t_0)}{\sqrt{N}}\mathbf{b}_0 + \mathbf{i}(x,t-t_0)$$

and therefore:

$$\mathbf{i}(x,t) = \frac{I(t)+I(t-t_0)}{\sqrt{N}}\mathbf{b}_0 + \frac{2}{L}\int_0^L d\xi [K^*(x,\xi,t)]\,\mathbf{v}^*(\xi) - \mathbf{i}(x,t-t_0)$$

N.B. In the particular case in which $\mathbf{v}^*$ is a piecewise constant function, we get:

$$\mathbf{v}^*(x) = \sum_{m=1}^{M} \mathbf{v}_m^* U(x; x_m, x_{m+1})$$

(with $x_1 = 0$ and $x_{M+1} = L$) and where:

$$U(x;a,b) = \begin{cases} 0, \text{ if} & x < a \\ 1, \text{ if} & a < x < b \\ 0, \text{ if} & x > b \end{cases}$$

We have:

I) $t < t_0$

$$\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{N}}\mathbf{b}_0 + 2\sum_{m=1}^{M}[K^{**}(x,t;x_m,x_{m+1})]\,\mathbf{v}_m^*$$

II) $t > t_0$

$$\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{N}}\mathbf{b}_0 + 2\sum_{m=1}^{M}[K^{**}(x,t;x_m,x_{m+1})]\,\mathbf{v}_m^* - 2\sum_{m=1}^{M}[K^{**}(x,t-t_0;x_m,x_{m+1})]\,\mathbf{v}_m^*$$

where (for $0 \le a < b \le L$):

$$[K^{**}(x,t;a,b)] = \frac{1}{L}\int_a^b d\xi\,[K^*(x,\xi,t)] = \frac{1}{L}\int_a^b d\xi \sum_{\substack{k=-(p-1)\\ k\neq 0}}^{p} \Gamma_k^*(x,\xi,t)\frac{\mathbf{b}_k\mathbf{b}_k^T}{\lambda_k} = \sum_{\substack{k=-(p-1)\\ k\neq 0}}^{p} \Gamma_k^{**}(x,t;a,b)\frac{\mathbf{b}_k\mathbf{b}_k^T}{\lambda_k}$$

$$\Gamma_k^{**}(x,t;a,b) = \frac{1}{L}\int_a^b d\xi\,\Gamma_k^*(x,\xi,t)$$

From the definition, we get:

$$\Gamma_k{}^*(x,\xi;t)=\Gamma_k{}^*(x,\xi;\infty)+\frac{L\lambda_k}{8}\sqrt{-\frac{\gamma_k}{r}}\sum_{n=-\infty}^{\infty}$$

$$\left[-e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right)-e^{|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right)+\right.$$

$$\left.+e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right)+e^{|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right)\right]$$

$$\Gamma_k{}^*(x,\xi;\infty)=L\lambda_k\sqrt{-\frac{\gamma_k}{r}}\left\{\frac{e^{-|x-\xi|\sqrt{-r\gamma_k}}-e^{-(x+\xi)\sqrt{-r\gamma_k}}}{4}-\frac{\sinh\left(x\sqrt{-r\gamma_k}\right)\sinh\left(\xi\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}\right\}$$

Therefore:

$$\Gamma_k^{**}(x,\infty;a,b)=\frac{1}{L}\int_a^b d\xi\,\Gamma_k^*(x,\xi;\infty)=$$
$$=\int_a^b d\xi\,\lambda_k\sqrt{-\frac{\gamma_k}{r}}\left\{\frac{e^{-|x-\xi|\sqrt{-r\gamma_k}}-e^{-(x+\xi)\sqrt{-r\gamma_k}}}{4}-\frac{\sinh\left(x\sqrt{-r\gamma_k}\right)\sinh\left(\xi\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}\right\}$$

Assuming $s^-=(\xi-x)\sqrt{-r\gamma_k}$, $s^+=(\xi+x)\sqrt{-r\gamma_k}$, $s=\xi\sqrt{-r\gamma_k}$ and thus $ds^-=ds^+=ds=d\xi\sqrt{-r\gamma_k}$, it result:

$$\Gamma_k^{**}(x,\infty;a,b)=\frac{\lambda_k}{4r}\int_{(a-x)\sqrt{-r\gamma_k}}^{(b-x)\sqrt{-r\gamma_k}}e^{-|s^-|}ds^- -\frac{\lambda_k}{4r}\int_{(a+x)\sqrt{-r\gamma_k}}^{(b+x)\sqrt{-r\gamma_k}}e^{-s^+}ds^+ -\frac{\lambda_k}{r}\frac{\sinh\left(x\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}\int_{a\sqrt{-r\gamma_k}}^{b\sqrt{-r\gamma_k}}\sinh(s)ds^+$$
$$=\frac{\lambda_k}{4r}\left[\operatorname{sgn}(t^-)\left(1-e^{-|s^-|}\right)\right]_{s^-=(a-x)\sqrt{-r\gamma_k}}^{s^-=(b-x)\sqrt{-r\gamma_k}}+\frac{\lambda_k}{4r}\left[e^{-s^+}\right]_{s^+=(a+x)\sqrt{-r\gamma_k}}^{s^+=(b+x)\sqrt{-r\gamma_k}}+$$
$$-\frac{\lambda_k}{r}\frac{\sinh\left(x\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}\left[\cosh(s)\right]_{s=a\sqrt{-r\gamma_k}}^{s=b\sqrt{-r\gamma_k}}$$
$$=\frac{\lambda_k}{4r}\left[\operatorname{sgn}(\xi-x)\left(1-e^{-|\xi-x|\sqrt{-r\gamma_k}}\right)\right]_{\xi=a}^{\xi=b}+\frac{\lambda_k}{4r}\left[e^{-(\xi+x)\sqrt{-r\gamma_k}}\right]_{\xi=a}^{\xi=b}+$$
$$-\frac{\lambda_k}{r}\frac{\sinh\left(x\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}\left[\cosh\left(\xi\sqrt{-r\gamma_k}\right)\right]_{\xi=a}^{\xi=b}$$

and finally [(o)]:

$$\boxed{\Gamma_k^{**}(x,\infty;a,b)=\frac{\lambda_k}{r}\left[\frac{\operatorname{sgn}(\xi-x)\left(1-e^{-|\xi-x|\sqrt{-r\gamma_k}}\right)+e^{-(\xi+x)\sqrt{-r\gamma_k}}}{4}-\frac{\sinh\left(x\sqrt{-r\gamma_k}\right)\cosh\left(\xi\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}\right]_{\xi=a}^{\xi=b}}$$

Regarding the transient term, it is:

(o) Note that $\int_0^q e^{-|u|}du=\left[\operatorname{sgn}(u)\left(1-e^{-|u|}\right)\right]_{u=0}^{u=q}$

$$\Gamma_k^{**}(x,t;a,b)=\Gamma_k^{**}(x,\infty;a,b)+\frac{\lambda_k}{8}\sqrt{-\frac{\gamma_k}{r}}\sum_{n=-\infty}^{\infty}\int_a^b d\xi$$

$$\left[-e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right)-e^{|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right)+\right.$$

$$\left.+e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right)+e^{|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right)\right]$$

Assuming $w^-=(x-\xi-2nL)\sqrt{-r\gamma_k}$, $w^+=(x+\xi-2nL)\sqrt{-r\gamma_k}$ and thus $-dw^-=dw^+=d\xi\sqrt{-r\gamma_k}$, it result:

$$\Gamma_k^{**}(x,t;a,b)-\Gamma_k^{**}(x,\infty;a,b)=\frac{\lambda_k}{8r}\sum_{n=-\infty}^{\infty}$$

$$\left[\int_{(x-a-2nL)\sqrt{-r\gamma_k}}^{(x-b-2nL)\sqrt{-r\gamma_k}}dw^-\left\{e^{-|w^-|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^-|\right)+e^{|w^-|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^-|\right)\right\}+\right.$$

$$\left.+\int_{(x+a-2nL)\sqrt{-r\gamma_k}}^{(x+b-2nL)\sqrt{-r\gamma_k}}dw^+\left\{e^{-|w^+|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^+|\right)+e^{|w^+|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^+|\right)\right\}\right]$$

Substitution of the relations

$$\int_0^q e^{-|u|}\operatorname{erfc}\left(\frac{1}{2\Omega}-\Omega|u|\right)du=\left[\operatorname{sgn}(u)\left\{\operatorname{erfc}\left(\frac{1}{2\Omega}\right)-e^{-|u|}\operatorname{erfc}\left(\frac{1}{2\Omega}-\Omega|u|\right)+e^{-\left(\frac{1}{2\Omega}\right)^2}\operatorname{erf}(\Omega|u|)\right\}\right]_{u=0}^{u=q}$$

$$\int_0^q e^{|u|}\operatorname{erfc}\left(\frac{1}{2\Omega}+\Omega|u|\right)du=\left[-\operatorname{sgn}(u)\left\{\operatorname{erfc}\left(\frac{1}{2\Omega}\right)-e^{|u|}\operatorname{erfc}\left(\frac{1}{2\Omega}+\Omega|u|\right)-e^{-\left(\frac{1}{2\Omega}\right)^2}\operatorname{erf}(\Omega|u|)\right\}\right]_{u=0}^{u=q}$$

(where $\Omega=\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}$) leads to:

$$\Gamma_k^{**}(x,t;a,b)-\Gamma_k^{**}(x,\infty;a,b)=\frac{\lambda_k}{8r}\sum_{n=-\infty}^{\infty}$$

$$\left[\operatorname{sgn}(w^-)\left\{\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}\right)-e^{-|w^-|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^-|\right)+e^{-\frac{rt}{\lambda_k}}\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^-|\right)\right\}+\right.$$

$$\left.-\operatorname{sgn}(w^-)\left\{\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}\right)-e^{|w^-|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^-|\right)-e^{-\frac{rt}{\lambda_k}}\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^-|\right)\right\}\right]_{w^-=(x-a-2nL)\sqrt{-r\gamma_k}}^{w^-=(x-b-2nL)\sqrt{-r\gamma_k}}$$

$$\left[\operatorname{sgn}(w^+)\left\{\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}\right)-e^{-|w^+|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^+|\right)+e^{-\frac{rt}{\lambda_k}}\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^+|\right)\right\}+\right.$$

$$\left.-\operatorname{sgn}(w^+)\left\{\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}\right)-e^{|w^+|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^+|\right)-e^{-\frac{rt}{\lambda_k}}\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^+|\right)\right\}\right]_{w^+=(x+a-2nL)\sqrt{-r\gamma_k}}^{w^+=(x+b-2nL)\sqrt{-r\gamma_k}}$$

Simplifying, it result:

$$\Gamma_k^{**}(x,t;a,b)-\Gamma_k^{**}(x,\infty;a,b)=$$

$$=\frac{\lambda_k}{8r}\sum_{n=-\infty}^{\infty}\left[\operatorname{sgn}(w^-)\left\{e^{|w^-|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^-|\right)-e^{-|w^-|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^-|\right)+\right.\right.$$

$$\left.\left.+2\,e^{-\frac{rt}{\lambda_k}}\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^-|\right)\right\}\right]_{w^-=(x-a-2nL)\sqrt{-r\gamma_k}}^{w^-=(x-b-2nL)\sqrt{-r\gamma_k}}+$$

$$+\left[\operatorname{sgn}(w^+)\left\{e^{|w^+|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^+|\right)-e^{-|w^+|}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^+|\right)+\right.\right.$$

$$\left.\left.+2\,e^{-\frac{rt}{\lambda_k}}\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{\lambda_k}{rt}}|w^+|\right)\right\}\right]_{w^+=(x-a-2nL)\sqrt{-r\gamma_k}}^{w^+=(x-b-2nL)\sqrt{-r\gamma_k}}$$

Substituting:

$$\Gamma_k^{**}(x,t;a,b)=\Gamma_k^{**}(x,\infty;a,b)+\frac{\lambda_k}{8r}\sum_{n=-\infty}^{\infty}$$

$$\Bigg[\operatorname{sgn}(x-\xi-2nL)\Bigg\{e^{|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)+$$

$$-e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)+2\,e^{-\frac{rt}{\lambda_k}}\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)\Bigg\}+$$

$$+\operatorname{sgn}(x+\xi-2nL)\Bigg\{e^{|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)+$$

$$-e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)+2\,e^{-\frac{rt}{\lambda_k}}\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)\Bigg\}\Bigg]_{\xi=a}^{\xi=b}$$

Taking into account that for convergence regions it's convenient to substitute erf = 1 − erfc, we get:

$$\Gamma_k^{**}(x,t;a,b)=\Gamma_k^{**}(x,\infty;a,b)+\frac{\lambda_k}{4r}e^{-\frac{rt}{\lambda_k}}\sum_{n=-\infty}^{\infty}\left[\operatorname{sgn}(x-\xi-2nL)+\operatorname{sgn}(x+\xi-2nL)\right]_{\xi=a}^{\xi=b}$$

$$+\frac{\lambda_k}{8r}\sum_{n=-\infty}^{\infty}$$

$$\Bigg[\operatorname{sgn}(x-\xi-2nL)\Bigg\{e^{|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)+$$

$$-e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)-2\,e^{-\frac{rt}{\lambda_k}}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)\Bigg\}+$$

$$+\operatorname{sgn}(x+\xi-2nL)\Bigg\{e^{|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)+$$

$$-e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)-2\,e^{-\frac{rt}{\lambda_k}}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)\Bigg\}\Bigg]_{\xi=a}^{\xi=b}$$

Breaking the first series in positive and negative terms, we have:

$$\frac{\lambda_k}{4r}e^{-\frac{rt}{\lambda_k}}\sum_{n=-\infty}^{\infty}\left[\operatorname{sgn}(x-\xi-2nL)+\operatorname{sgn}(x+\xi-2nL)\right]_{\xi=a}^{\xi=b}=\frac{\lambda_k}{4r}e^{-\frac{rt}{\lambda_k}}\left[\operatorname{sgn}(x-\xi)+\operatorname{sgn}(x+\xi)\right]_{\xi=a}^{\xi=b}+$$

$$+\frac{\lambda_k}{4r}e^{-\frac{rt}{\lambda_k}}\sum_{n=1}^{\infty}\left[\operatorname{sgn}(x-\xi-2nL)+\operatorname{sgn}(x+\xi-2nL)+\operatorname{sgn}(x-\xi+2nL)+\operatorname{sgn}(x+\xi+2nL)\right]_{\xi=a}^{\xi=b}=$$

$$=\frac{\lambda_k}{4r}e^{-\frac{rt}{\lambda_k}}\left[\operatorname{sgn}(x-\xi)+1\right]_{\xi=a}^{\xi=b}=\frac{\lambda_k}{4r}e^{-\frac{rt}{\lambda_k}}\left[\operatorname{sgn}(x-\xi)\right]_{\xi=a}^{\xi=b}$$

Note that, being $n > 1$, $0 < x < L$ and $0 < \xi < L$, it result $x - \xi - 2nL < 0$, $x + \xi - 2nL < 0$, $x - \xi + 2nL > 0$, $x + \xi + 2nL > 0$ and thus the sign series is null. Substitution leads to:

$$\Gamma_k^{**}(x,t;a,b)=\Gamma_k^{**}(x,\infty;a,b)+\frac{\lambda_k}{4r}e^{-\frac{rt}{\lambda_k}}\left[\operatorname{sgn}(x-\xi)\right]_{\xi=a}^{\xi=b}+\frac{\lambda_k}{8r}\sum_{n=-\infty}^{\infty}$$

$$\left[\operatorname{sgn}(x-\xi-2nL)\left\{e^{|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)+\right.\right.$$

$$\left.-e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)-2e^{-\frac{rt}{\lambda_k}}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)\right\}+$$

$$+\operatorname{sgn}(x+\xi-2nL)\left\{e^{|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)+\right.$$

$$\left.\left.-e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)-2e^{-\frac{rt}{\lambda_k}}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)\right\}\right]_{\xi=a}^{\xi=b}$$

Note that the erfc functions in the series do not tend a 0, for increasing n, since the argument is negative and decreasing. In order to improve the convergence the following substitution can be done: erfc(–x) = 1 – erf(–x) = 1 + erf(x) = 2 – erfc(x). The remaining series can be treated separately:

$$\Gamma_k^{**}(x,t;a,b)=\Gamma_k^{**}(x,\infty;a,b)+\frac{\lambda_k}{4r}e^{-\frac{rt}{\lambda_k}}\left[\operatorname{sgn}(x-\xi)\right]_{\xi=a}^{\xi=b}+$$

$$-\frac{\lambda_k}{4r}\sum_{n=-\infty}^{\infty}\left[\operatorname{sgn}(x-\xi-2nL)e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}+\operatorname{sgn}(x+\xi-2nL)e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\right]_{\xi=a}^{\xi=b}+$$

$$+\frac{\lambda_k}{8r}\sum_{n=-\infty}^{\infty}$$

$$\left[\operatorname{sgn}(x-\xi-2nL)\left\{e^{|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)+\right.\right.$$

$$\left.+e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(-\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)-2e^{-\frac{rt}{\lambda_k}}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)\right\}+$$

$$+\operatorname{sgn}(x+\xi-2nL)\left\{e^{|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)+\right.$$

$$\left.\left.+e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(-\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)-2e^{-\frac{rt}{\lambda_k}}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)\right\}\right]_{\xi=a}^{\xi=b}$$

In fact:

$$-\frac{\lambda_k}{4r}\sum_{n=-\infty}^{\infty}\left[\operatorname{sgn}(x-\xi-2nL)e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}+\operatorname{sgn}(x+\xi-2nL)e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\right]=$$

$$=-\frac{\lambda_k}{4r}\left[\operatorname{sgn}(x-\xi)e^{-|x-\xi|\sqrt{-r\gamma_k}}+\operatorname{sgn}(x+\xi)e^{-|x+\xi|\sqrt{-r\gamma_k}}\right]-\frac{\lambda_k}{4r}\sum_{n=1}^{\infty}$$

$$\left[\operatorname{sgn}(x-\xi-2nL)e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}+\operatorname{sgn}(x+\xi-2nL)e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}+\right.$$

$$\left.+\operatorname{sgn}(x-\xi+2nL)e^{-|x-\xi+2nL|\sqrt{-r\gamma_k}}+\operatorname{sgn}(x+\xi+2nL)e^{-|x+\xi+2nL|\sqrt{-r\gamma_k}}\right]=$$

$$=-\frac{\lambda_k}{4r}\left[\operatorname{sgn}(x-\xi)e^{-|x-\xi|\sqrt{-r\gamma_k}}+e^{-(x+\xi)\sqrt{-r\gamma_k}}\right]-\frac{\lambda_k}{4r}\sum_{n=1}^{\infty}$$

$$\left[-e^{(x-\xi-2nL)\sqrt{-r\gamma_k}}-e^{(x+\xi-2nL)\sqrt{-r\gamma_k}}+e^{-(x-\xi+2nL)\sqrt{-r\gamma_k}}+e^{-(x+\xi+2nL)\sqrt{-r\gamma_k}}\right]=$$

$$=-\frac{\lambda_k}{4r}\left[\operatorname{sgn}(x-\xi)e^{-|x-\xi|\sqrt{-r\gamma_k}}+e^{-(x+\xi)\sqrt{-r\gamma_k}}\right]-\frac{\lambda_k}{4r}\sum_{n=1}^{\infty}e^{-2nL\sqrt{-r\gamma_k}}$$

$$\left[-e^{(x-\xi)\sqrt{-r\gamma_k}}-e^{(x+\xi)\sqrt{-r\gamma_k}}+e^{-(x-\xi)\sqrt{-r\gamma_k}}+e^{-(x+\xi)\sqrt{-r\gamma_k}}\right]=$$

$$=-\frac{\lambda_k}{4r}\left[\operatorname{sgn}(x-\xi)e^{-|x-\xi|\sqrt{-r\gamma_k}}+e^{-(x+\xi)\sqrt{-r\gamma_k}}\right]-\frac{\lambda_k}{4r}\frac{1}{e^{2L\sqrt{-r\gamma_k}}-1}$$

$$\left[-2e^{x\sqrt{-r\gamma_k}}\cosh\left(\xi\sqrt{-r\gamma_k}\right)+2e^{-x\sqrt{-r\gamma_k}}\cosh\left(\xi\sqrt{-r\gamma_k}\right)\right]=$$

$$=-\frac{\lambda_k}{4r}\left[\operatorname{sgn}(x-\xi)e^{-|x-\xi|\sqrt{-r\gamma_k}}+e^{-(x+\xi)\sqrt{-r\gamma_k}}\right]+\frac{\lambda_k}{r}\frac{\sinh\left(x\sqrt{-r\gamma_k}\right)\cosh\left(\xi\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}$$

Therefore, finally:

$$\Gamma_k^{**}(x,t;a,b)=\Gamma_k^{**}(x,\infty;a,b)+$$

$$\left[-\frac{\lambda_k}{4r}\operatorname{sgn}(x-\xi)\left(e^{-|x-\xi|\sqrt{-r\gamma_k}}-e^{-\frac{rt}{\lambda_k}}\right)-\frac{\lambda_k}{4r}e^{-(x+\xi)\sqrt{-r\gamma_k}}+\frac{\lambda_k}{r}\frac{\sinh\left(x\sqrt{-r\gamma_k}\right)\cosh\left(\xi\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}\right]_{\xi=a}^{\xi=b}+$$

$$+\frac{\lambda_k}{8r}\sum_{n=-\infty}^{\infty}\left[\operatorname{sgn}(x-\xi-2nL)\left\{e^{|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)+\right.\right.$$

$$\left.+e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(-\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)-2e^{-\frac{rt}{\lambda_k}}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right)\right\}+$$

$$+\operatorname{sgn}(x+\xi-2nL)\left\{e^{|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)+\right.$$

$$\left.\left.+e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(-\sqrt{\frac{rt}{\lambda_k}}+\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)-2e^{-\frac{rt}{\lambda_k}}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)\right\}\right]_{\xi=a}^{\xi=b}$$

8. Perfect Conductor Case (r = 0)

In the particular case in which $r \to 0$, it's possible to specialize the form of Γ* and of Γ** as follow:

$$\Gamma_k{}^*(x,\xi;\infty) = L\lambda_k\sqrt{-\frac{\gamma_k}{r}}\left\{\frac{e^{-|x-\xi|\sqrt{-r\gamma_k}} - e^{-(x+\xi)\sqrt{-r\gamma_k}}}{4} - \frac{\sinh\left(x\sqrt{-r\gamma_k}\right)\sinh\left(\xi\sqrt{-r\gamma_k}\right)}{e^{2L\sqrt{-r\gamma_k}}-1}\right\}$$

$$\overset{r\to 0}{=} L\lambda_k\sqrt{-\frac{\gamma_k}{r}}\left\{\frac{1-|x-\xi|\sqrt{-r\gamma_k} - 1 + (x+\xi)\sqrt{-r\gamma_k}}{4} - \frac{x\sqrt{-r\gamma_k}\,\xi\sqrt{-r\gamma_k}}{1+2L\sqrt{-r\gamma_k}-1}\right\}$$

$$= -L\lambda_k\gamma_k\left\{\frac{(x+\xi)-|x-\xi|}{4} - \frac{x\xi}{2L}\right\}$$

Thus:

$$\boxed{\Gamma_k{}^*(x,\xi;\infty) \overset{r\to 0}{=} -\frac{L\lambda_k\gamma_k}{2}\left\{\frac{(x+\xi)-|x-\xi|}{2} - \frac{x\xi}{L}\right\}}$$

Moreover:

$$\Gamma_k{}^*(x,\xi;t) - \Gamma_k{}^*(x,\xi;\infty) = \frac{L\lambda_k}{8}\sqrt{-\frac{\gamma_k}{r}}\sum_{n=-\infty}^{\infty}$$

$$\left[-e^{-|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right) - e^{|x-\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} + \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right)+\right.$$

$$\left.+e^{-|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} - \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right) + e^{|x+\xi-2nL|\sqrt{-r\gamma_k}}\operatorname{erfc}\left(\sqrt{\frac{rt}{\lambda_k}} + \sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right)\right]$$

thus

$$\Gamma_k{}^*(x,\xi;t) - \Gamma_k{}^*(x,\xi;\infty) \overset{r\to 0}{=} \frac{L\lambda_k}{8}\sqrt{-\frac{\gamma_k}{r}}\sum_{n=-\infty}^{\infty}$$

$$\left[\left(-1+|x-\xi-2nL|\sqrt{-r\gamma_k}\right)\left(1+\operatorname{erf}\left(\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right) - \frac{2}{\sqrt{\pi}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2}\sqrt{\frac{rt}{\lambda_k}}\right)+\right.$$

$$\left(-1-|x-\xi-2nL|\sqrt{-r\gamma_k}\right)\left(1-\operatorname{erf}\left(\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right) - \frac{2}{\sqrt{\pi}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2}\sqrt{\frac{rt}{\lambda_k}}\right)+$$

$$+\left(1-|x+\xi-2nL|\sqrt{-r\gamma_k}\right)\left(1+\operatorname{erf}\left(\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right) - \frac{2}{\sqrt{\pi}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2}\sqrt{\frac{rt}{\lambda_k}}\right)+$$

$$\left.+\left(1+|x+\xi-2nL|\sqrt{-r\gamma_k}\right)\left(1-\operatorname{erf}\left(\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right) - \frac{2}{\sqrt{\pi}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2}\sqrt{\frac{rt}{\lambda_k}}\right)\right]$$

$$= \frac{L\lambda_k}{8}\sqrt{-\frac{\gamma_k}{r}}\sum_{n=-\infty}^{\infty} 2\sqrt{-r\gamma_k}$$

$$\left[|x-\xi-2nL|\,\mathrm{erf}\left(\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x-\xi}{2}-nL\right|\right) - |x+\xi-2nL|\,\mathrm{erf}\left(\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|\frac{x+\xi}{2}-nL\right|\right) + \right.$$

$$\left. + \frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}\left(e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2} - e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2} \right) \right]$$

and finally:

$$\Gamma_k{}^*(x,\xi;t) - \Gamma_k{}^*(x,\xi;\infty) \overset{r\to 0}{=} -\frac{L\gamma_k\lambda_k}{4}\sum_{n=-\infty}^{\infty}$$

$$\left[|x-\xi-2nL|\,\mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right) - |x+\xi-2nL|\,\mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right) + \right.$$

$$\left. + \frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}\left(e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2} - e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2} \right) \right]$$

Note that while for the last term of this series the convergence is assured for n → ±∞, the sum of the first two terms is semi-convergent since goes to ±2ξ for n → ±∞. In order to improve the convergence for n → ±∞ the following the operations are performed:

1° the substitution erf(x) = 1 − erfc(x)

2° the separation of the series from − ∞ to +∞ in two series (changing sign to the negative n):

$$\Gamma_k{}^*(x,\xi;t) - \Gamma_k{}^*(x,\xi;\infty) \overset{r\to 0}{=} -\frac{L\gamma_k\lambda_k}{4}\sum_{n=-\infty}^{\infty}\left[|x-\xi-2nL| - |x+\xi-2nL|\right] - \frac{L\gamma_k\lambda_k}{4}\sum_{n=-\infty}^{\infty}$$

$$\left. -|x-\xi-2nL|\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi-2nL|\right) + |x+\xi-2nL|\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi-2nL|\right)\right]$$

$$-\frac{L\gamma_k\lambda_k}{4}\sum_{n=-\infty}^{\infty}\left[\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}\left(e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2} - e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2} \right)\right]$$

$$= -\frac{L\gamma_k\lambda_k}{4}\left\{|x-\xi| - |x+\xi| + \sum_{n=1}^{\infty}\left[|x-\xi-2nL| - |x+\xi-2nL| + |x-\xi+2nL| - |x+\xi+2nL|\right]\right\} +$$

$$+ \frac{L\gamma_k\lambda_k}{4}\left\{|x-\xi|\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi|\right) - |x+\xi|\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi|\right)\right\} +$$

$$-\frac{L\gamma_k\lambda_k}{4}\sum_{n=1}^{\infty}\left[-|x-\xi+2nL|\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x-\xi+2nL|\right) + |x+\xi+2nL|\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}|x+\xi+2nL|\right)\right.$$

$$-\left|x-\xi-2nL\right|\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|x-\xi-2nL\right|\right)+\left|x+\xi-2nL\right|\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|x+\xi-2nL\right|\right)\Bigg]\Bigg]$$

$$-\frac{L\gamma_k\lambda_k}{4}\sum_{n=-\infty}^{\infty}\left[\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}\left(e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2}-e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2}\right)\right]\Bigg]$$

Note that, being $n > 1$, $0 < x < L$ and $0 < \xi < L$ it result $x - \xi - 2nL < 0$, $x + \xi - 2nL < 0$, $x - \xi + 2nL > 0$, $x + \xi + 2nL > 0$ and thus the first series is null. Moreover, the separation of the last series, evidencing the zero-th order terms e simplifying, gives:

$$\Gamma_k^*(x,\xi;t)-\Gamma_k^*(x,\xi;\infty)\overset{r\to 0}{=}$$

$$=-\frac{L\gamma_k\lambda_k}{4}\left\{\left|x-\xi\right|\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|x-\xi\right|\right)+\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}\right)^2}+\right.$$

$$\left.-\left|x+\xi\right|\operatorname{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|x+\xi\right|\right)-\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}\right)^2}\right\}+$$

$$-\frac{L\gamma_k\lambda_k}{4}\sum_{n=1}^{\infty}\left[-\left|x-\xi+2nL\right|\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|x-\xi+2nL\right|\right)+\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}+nL\right)^2}+\right.$$

$$+\left|x+\xi+2nL\right|\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|x+\xi+2nL\right|\right)-\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}+nL\right)^2}+$$

$$-\left|x-\xi-2nL\right|\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|x-\xi-2nL\right|\right)+\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2}+$$

$$\left.+\left|x+\xi-2nL\right|\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}\left|x+\xi-2nL\right|\right)-\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2}\right]$$

It's now possible to eliminate all the modules thanks to the symmetry of the erf function and being $n > 1$, $0 < x < L$ and $0 < \xi < L$, so that it result $x - \xi - 2nL < 0$, $x + \xi - 2nL < 0$, $x - \xi + 2nL > 0$, $x + \xi + 2nL > 0$ e $x + \xi > 0$:

$$
\begin{aligned}
&\Gamma_k{}^*(x,\xi;t)-\Gamma_k{}^*(x,\xi;\infty)\overset{r\to 0}{=}\\
&=-\frac{L\gamma_k\lambda_k}{4}\Bigg\{(x-\xi)\,\mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-\xi)\right)+\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}\right)^2}+\\
&\qquad-(x+\xi)\,\mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+\xi)\right)-\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}\right)^2}\Bigg\}+\\
&-\frac{L\gamma_k\lambda_k}{4}\sum_{n=1}^{\infty}\Bigg[-(x-\xi+2nL)\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-\xi+2nL)\right)+\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}+nL\right)^2}+\\
&\qquad+(x+\xi+2nL)\,\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+\xi+2nL)\right)-\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}+nL\right)^2}+\\
&\qquad+(x-\xi-2nL)\,\mathrm{erfc}\left(-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-\xi-2nL)\right)+\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x-\xi}{2}-nL\right)^2}+\\
&\qquad-(x+\xi-2nL)\,\mathrm{erfc}\left(-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+\xi-2nL)\right)-\frac{2}{\sqrt{\pi}}\sqrt{\frac{t}{-\lambda_k\gamma_k}}e^{\frac{\lambda_k\gamma_k}{t}\left(\frac{x+\xi}{2}-nL\right)^2}\Bigg]
\end{aligned}
$$

Regarding Γ**, from the definition it result:

$$\Gamma_k^{**}(x,t;a,b)=\frac{1}{L}\int_a^b d\xi\,\Gamma_k^*(x,\xi,t)$$

thus (*), assigning $\sigma=(\xi-x)$ and $d\sigma=d\xi$, it result:

$$\Gamma_k^{**}(x,\infty;a,b)\overset{r\to 0}{=}-\frac{\lambda_k\gamma_k}{2}\left\{\left[\frac{(x+\xi)^2}{4}-\frac{x\xi^2}{2L}\right]_{\xi=a}^{\xi=b}-\frac{1}{2}\int_{(a-x)}^{(b-x)}|\sigma|\,d\sigma\right\}$$

and finally:

$$\Gamma_k^{**}(x,\infty;a,b)\overset{r\to 0}{=}-\frac{\lambda_k\gamma_k}{4}\left[\frac{(x+\xi)^2-\mathrm{sgn}(\xi-x)(x-\xi)^2}{2}-\frac{x\xi^2}{L}\right]_{\xi=a}^{\xi=b}$$

for what concerns the transient term, assigning $\omega_-=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(\theta_x x-\xi+\theta_L 2nL)$, $\omega_+=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(\theta_x x+\xi+\theta_L 2nL)$, with $\theta_x,\theta_L = 0, \pm 1$ and $-d\omega_-=d\omega_+=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}d\xi$ it result:

(*)Note that $\int_0^q |u|\,du=\left[\mathrm{sgn}(u)\left(\frac{u^2}{2}\right)\right]_{u=0}^{u=q}$

$$
\Gamma_k^{**}(x,t;a,b)-\Gamma_k^{**}(x,\infty;a,b)\overset{r\to 0}{=}
$$

$$
-\frac{\gamma_k\lambda_k}{4}\left\{\frac{4t}{\gamma_k\lambda_k}\int_{\bar{a}_-}^{\bar{b}_-}\omega_-\,\mathrm{erf}(\omega_-)\,d\omega_-+\frac{2}{\sqrt{\pi}}\frac{2t}{\gamma_k\lambda_k}\int_{\bar{a}_-}^{\bar{b}_-}e^{-(\omega_-)^2}\,d\omega_-+\right.
$$

$$
\left.+\frac{4t}{\gamma_k\lambda_k}\int_{\bar{a}_+}^{\bar{b}_+}\omega_+\,\mathrm{erf}(\omega_+)\,d\omega_++\frac{2}{\sqrt{\pi}}\frac{2t}{\gamma_k\lambda_k}\int_{\bar{a}_+}^{\bar{b}_+}e^{-(\omega_+)^2}\,d\omega_+\right\}+
$$

$$
-\frac{\gamma_k\lambda_k}{4}\sum_{n=1}^{\infty}\left[-\frac{4t}{\gamma_k\lambda_k}\int_{\hat{a}_-}^{\hat{b}_-}\omega_-\,\mathrm{erfc}(\omega_-)\,d\omega_-+\frac{2}{\sqrt{\pi}}\frac{2t}{\gamma_k\lambda_k}\int_{\hat{a}_-}^{\hat{b}_-}e^{-(\omega_-)^2}\,d\omega_-+\right.
$$

$$
-\frac{4t}{\gamma_k\lambda_k}\int_{\hat{a}_+}^{\hat{b}_+}\omega_+\,\mathrm{erfc}(\omega_+)\,d\omega_++\frac{2}{\sqrt{\pi}}\frac{2t}{\gamma_k\lambda_k}\int_{\hat{a}_+}^{\hat{b}_+}e^{-(\omega_+)^2}\,d\omega_++
$$

$$
+\frac{4t}{\gamma_k\lambda_k}\int_{\breve{a}_-}^{\breve{b}_-}\omega_-\,\mathrm{erfc}(\omega_-)\,d\omega_--\frac{2}{\sqrt{\pi}}\frac{2t}{\gamma_k\lambda_k}\int_{\breve{a}_-}^{\breve{b}_-}e^{-(\omega_-)^2}\,d\omega_-+
$$

$$
\left.+\frac{4t}{\gamma_k\lambda_k}\int_{\breve{a}_+}^{\breve{b}_+}\omega_+\,\mathrm{erfc}(\omega_+)\,d\omega_+-\frac{2}{\sqrt{\pi}}\frac{2t}{\gamma_k\lambda_k}\int_{\breve{a}_+}^{\breve{b}_+}e^{-(\omega_+)^2}\,d\omega_+\right]
$$

where

$$
\bar{a}_-=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-a),\ \bar{b}_-=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-b),
$$

$$
\bar{a}_+=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+a),\ \bar{b}_+=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+b)
$$

$$
\hat{a}_-=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-a+2nL),\ \hat{b}_-=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-b+2nL),
$$

$$
\hat{a}_+=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+a+2nL),\ \hat{b}_+=\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+b+2nL)
$$

$$
\breve{a}_-=-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+a-2nL),\ \breve{b}_-=-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+b-2nL),
$$

$$
\breve{a}_+=-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-a-2nL),\ \breve{b}_+=-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-b-2nL)
$$

Simplifying e taking into account the relations

$$
2\int_0^q u\,\mathrm{erf}(u)du+\frac{2}{\sqrt{\pi}}\int_0^q e^{-u^2}du=\left[\left(u^2+\frac{1}{2}\right)\mathrm{erf}(u)+\frac{u}{\sqrt{\pi}}e^{-u^2}\right]_{u=0}^{u=q}
$$

$$
2\int_0^q u\,\mathrm{erfc}(u)du-\frac{2}{\sqrt{\pi}}\int_0^q e^{-u^2}du=\left[\left(u^2+\frac{1}{2}\right)\mathrm{erfc}(u)-\frac{u}{\sqrt{\pi}}e^{-u^2}\right]_{u=0}^{u=q}
$$

we get:

$$
\Gamma_k^{**}(x,t;a,b)-\Gamma_k^{**}(x,\infty;a,b)\overset{r\to 0}{=}
$$

$$
-\frac{t}{2}\left\{\left[\left((\omega_-)^2+\frac{1}{2}\right)\mathrm{erf}(\omega_-)+\frac{\omega_-}{\sqrt{\pi}}e^{-(\omega_-)^2}\right]_{\ddot{a}_-}^{\bar{b}_-}+\left[\left((\omega_+)^2+\frac{1}{2}\right)\mathrm{erf}(\omega_+)+\frac{\omega_+}{\sqrt{\pi}}e^{-(\omega_+)^2}\right]_{\ddot{a}_+}^{\bar{b}_+}\right\}+
$$

$$
-\frac{t}{2}\sum_{n=1}^{\infty}\left\{-\left[\left((\omega_-)^2+\frac{1}{2}\right)\mathrm{erfc}(\omega_-)-\frac{\omega_-}{\sqrt{\pi}}e^{-(\omega_-)^2}\right]_{\widehat{a}_-}^{\widehat{b}_-}-\left[\left((\omega_+)^2+\frac{1}{2}\right)\mathrm{erfc}(\omega_+)-\frac{\omega_+}{\sqrt{\pi}}e^{-(\omega_+)^2}\right]_{\widehat{a}_+}^{\widehat{b}_+}+\right.
$$

$$
\left.+\left[\left((\omega_+)^2+\frac{1}{2}\right)\mathrm{erfc}(\omega_+)-\frac{\omega_+}{\sqrt{\pi}}e^{-(\omega_+)^2}\right]_{\breve{a}_+}^{\breve{b}_+}+\left[\left((\omega_-)^2+\frac{1}{2}\right)\mathrm{erfc}(\omega_-)-\frac{\omega_-}{\sqrt{\pi}}e^{-(\omega_-)^2}\right]_{\breve{a}_-}^{\breve{b}_-}\right\}
$$

and finally:

$$
\Gamma_k^{**}(x,t;a,b)-\Gamma_k^{**}(x,\infty;a,b)\overset{r\to 0}{=}
$$

$$
-\frac{t}{4}\left[\left(1-\frac{\lambda_k\gamma_k}{2t}(x-\xi)^2\right)\mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-\xi)\right)+\sqrt{\frac{-\lambda_k\gamma_k}{\pi t}}(x-\xi)e^{\frac{\lambda_k\gamma_k}{4t}(x-\xi)^2}+\right.
$$

$$
\left.+\left(1-\frac{\lambda_k\gamma_k}{2t}(x+\xi)^2\right)\mathrm{erf}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+\xi)\right)+\sqrt{\frac{-\lambda_k\gamma_k}{\pi t}}(x+\xi)e^{\frac{\lambda_k\gamma_k}{4t}(x+\xi)^2}\right]_{\xi=a}^{\xi=b}-\frac{t}{4}\sum_{n=1}^{\infty}
$$

$$
\left[-\left(1-\frac{\lambda_k\gamma_k}{2t}(x-\xi+2nL)^2\right)\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-\xi+2nL)\right)+\sqrt{\frac{-\lambda_k\gamma_k}{\pi t}}(x-\xi+2nL)e^{\frac{\lambda_k\gamma_k}{4t}(x-\xi+2nL)^2}+\right.
$$

$$
-\left(1-\frac{\lambda_k\gamma_k}{2t}(x+\xi+2nL)^2\right)\mathrm{erfc}\left(\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+\xi+2nL)\right)+\sqrt{\frac{-\lambda_k\gamma_k}{\pi t}}(x+\xi+2nL)e^{\frac{\lambda_k\gamma_k}{4t}(x+\xi+2nL)^2}+
$$

$$
+\left(1-\frac{\lambda_k\gamma_k}{2t}(x-\xi-2nL)^2\right)\mathrm{erfc}\left(-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x-\xi-2nL)\right)+\sqrt{\frac{-\lambda_k\gamma_k}{\pi t}}(x-\xi-2nL)e^{\frac{\lambda_k\gamma_k}{4t}(x-\xi-2nL)^2}+
$$

$$
\left.+\left(1-\frac{\lambda_k\gamma_k}{2t}(x+\xi-2nL)^2\right)\mathrm{erfc}\left(-\frac{1}{2}\sqrt{\frac{-\lambda_k\gamma_k}{t}}(x+\xi-2nL)\right)+\sqrt{\frac{-\lambda_k\gamma_k}{\pi t}}(x+\xi-2nL)e^{\frac{\lambda_k\gamma_k}{4t}(x+\xi-2nL)^2}\right]_{\xi=a}^{\xi=b}
$$

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{1} I.S. Gradshteyn, I.M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press: London, 1980, p. 1112.

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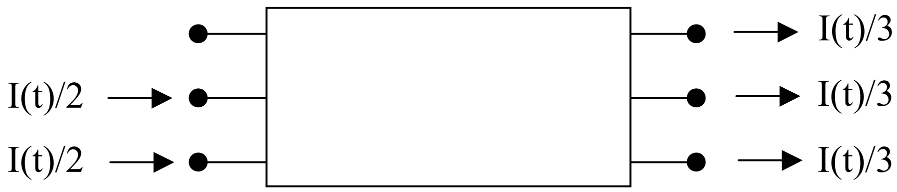
APPENDIX A – FORCED CURRENT DISTRIBUTION IN A 3-STRAND CABLE

Consider, as shown in the figure, a 3-strand cable with an insulated input strand. Moreover, assume that the flowing current is equally distributed on the two input strand and on the three output strands.



The longitudinal resistance (r) ant the external voltage (**v**) are assumed to be zero. Consequently, the currents flowing in the 3-strand cable are described by the following system:

$$
(P)\quad \begin{cases} [G][M]\dfrac{\partial \mathbf{i}}{\partial t}(x,t) + \dfrac{\partial^2 \mathbf{i}}{\partial x^2}(x,t) = 0 \\ \mathbf{i}(x, t=0) = 0 \\ i_1(x=0,t) = 0, \quad i_2(x=0,t) = i_3(x=0,t) = \dfrac{I(t)}{2} \\ i_1(x=L,t) = i_2(x=L,t) = i_3(x=L,t) = \dfrac{I(t)}{3} \end{cases} \qquad \text{and } \mathbf{i}, \mathbf{v} \in R^3
$$

with I(0) = 0 and where [M] and [G] are the following constant-coefficient circulant symmetric matrixes:

$$
[M] = \begin{bmatrix} m_{11} & m_{12} & m_{12} \\ m_{12} & m_{11} & m_{12} \\ m_{12} & m_{12} & m_{11} \end{bmatrix} \qquad [G] = \begin{bmatrix} -2g_{12} & g_{12} & g_{12} \\ g_{12} & -2g_{12} & g_{12} \\ g_{12} & g_{12} & -2g_{12} \end{bmatrix}
$$

Defining the orthonormal spectral basis $\mathbf{b}_k$, with k = 1, 0, –1, that it's the same for [M] and [G]:

$$
\mathbf{b}_0 = \frac{1}{\sqrt{3}}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad \mathbf{b}_1 = \sqrt{\frac{2}{3}}\begin{Bmatrix} 1 \\ -1/2 \\ -1/2 \end{Bmatrix} \qquad \mathbf{b}_{-1} = \frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix}
$$

the matrixes [M] and [G] can be written as

$$
[M] = \lambda_0 \mathbf{b}_0 \mathbf{b}_0^T + \lambda_1 \mathbf{b}_1 \mathbf{b}_1^T + \lambda_{-1} \mathbf{b}_{-1} \mathbf{b}_{-1}^T \qquad [G] = \gamma_0 \mathbf{b}_0 \mathbf{b}_0^T + \gamma_1 \mathbf{b}_1 \mathbf{b}_1^T + \gamma_{-1} \mathbf{b}_{-1} \mathbf{b}_{-1}^T
$$

with eigenvalues given by

$$
\lambda_0 = m_{11} + 2m_{12} \qquad \gamma_0 = 0
$$

$$
\lambda_1 = \lambda_{-1} = m_{11} - m_{12} \qquad \gamma_1 = \gamma_{-1} = -3g_{12}
$$

Decomposing **i** as follows

$$
\mathbf{i}(x,t) = \eta_0(x,t)\mathbf{b}_0 + \eta_1(x,t)\mathbf{b}_1 + \eta_{-1}(x,t)\mathbf{b}_{-1}
$$

Problem (P) is simply divided in the following three problems:

$$
(P_0)\quad \begin{cases} \dfrac{\partial^2 \eta_0}{\partial x^2}(x,t) = 0 \\ \eta_0(x, t=0) = 0, \quad \eta_0(x=0,t) = \dfrac{I(t)}{\sqrt{3}}, \quad \eta_0(x=L,t) = \dfrac{I(t)}{\sqrt{3}} \end{cases}
$$

$$(P_{-1})\ \begin{cases} \gamma_{-1}\lambda_{-1}\dfrac{\partial\eta_{-1}}{\partial t}(x,t)+\dfrac{\partial^2\eta_{-1}}{\partial x^2}(x,t)=0 \\ \eta_{-1}(x,t=0)=0, \quad \eta_{-1}(x=0,t)=0, \quad \eta_{-1}(x=L,t)=0 \end{cases}$$

$$(P_1)\ \begin{cases} \gamma_1\lambda_1\dfrac{\partial\eta_1}{\partial t}(x,t)+\dfrac{\partial^2\eta_1}{\partial x^2}(x,t)=0 \\ \eta_1(x,t=0)=0, \quad \eta_1(x=0,t)=-\dfrac{I(t)}{\sqrt{6}}, \quad \eta_1(x=L,t)=0 \end{cases}$$

Problems (P_0) and (P_{-1}) have the simple solutions

$$\eta_0(x,t)=\frac{I(t)}{\sqrt{3}} \qquad\qquad \eta_{-1}(x,t)=0$$

While problem (P_1) requires a different approach. Defining

$$\varphi(x,t)=\eta_1(x,t)+\frac{I(t)}{\sqrt{6}}\left(1-\frac{x}{L}\right)$$

gives

$$(P_\varphi)\ \begin{cases} \gamma_1\lambda_1\dfrac{\partial\varphi}{\partial t}(x,t)+\dfrac{\partial^2\varphi}{\partial x^2}(x,t)=\gamma_1 u(x,t) \\ \eta_1(x,t=0)=0, \quad \eta_1(x=0,t)=0, \quad \eta_1(x=L,t)=0 \end{cases}$$

which has been already treated and gives (having defined $u(x,t)=\lambda_1\dfrac{I'(t)}{\sqrt{6}}\left(1-\dfrac{x}{L}\right)$ where the prime denotes differentiation with respect to the argument)

$$\varphi(x,t)=\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\,\frac{u(\xi,\tau)}{\lambda_1}\Gamma_1(x,\xi,t-\tau)$$

Thus

$$\eta_1(x,t)=-\frac{I(t)}{\sqrt{6}}\left(1-\frac{x}{L}\right)+\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\,\frac{I'(\tau)}{\sqrt{6}}\left(1-\frac{\xi}{L}\right)\Gamma_1(x,\xi,t-\tau)$$

and the currents are

$$\mathbf{i}(x,t)=\frac{I(t)}{\sqrt{3}}\mathbf{b}_0+\left[-\frac{I(t)}{\sqrt{6}}\left(1-\frac{x}{L}\right)+\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\,\frac{I'(\tau)}{\sqrt{6}}\left(1-\frac{\xi}{L}\right)\Gamma_1(x,\xi,t-\tau)\right]\mathbf{b}_1$$

In particular, for a current ramp

$$I(t)=\begin{cases} 0, \text{ if } & t<0 \\ I_0 t/t_0, \text{ if } & 0<t<t_0 \\ I_0, \text{ if } & t>t_0 \end{cases} \quad\Rightarrow\quad I'(t)=\begin{cases} 0, \text{ if } & t<0 \\ I_0/t_0, \text{ if } & 0<t<t_0 \\ 0, \text{ if } & t>t_0 \end{cases}$$

this becomes

$$
\mathbf{i}(x,t)=\begin{cases}0, & \text{if } t<0\\[4pt] \dfrac{\alpha t}{\sqrt{3}}\mathbf{b}_0+\left[-\dfrac{\alpha t}{\sqrt{6}}\left(1-\dfrac{x}{L}\right)+\dfrac{\alpha}{\sqrt{6}}\dfrac{2}{L}\displaystyle\int_0^L d\xi\left(1-\dfrac{\xi}{L}\right)\Gamma_1^*(x,\xi,t)\right]\mathbf{b}_1, & \text{if } 0<t<t_0\\[4pt] \dfrac{I_0}{\sqrt{3}}\mathbf{b}_0+\left[-\dfrac{I_0}{\sqrt{6}}\left(1-\dfrac{x}{L}\right)+\dfrac{\alpha}{\sqrt{6}}\dfrac{2}{L}\displaystyle\int_0^L d\xi\left(1-\dfrac{\xi}{L}\right)\left\{\Gamma_1^*(x,\xi,t)-\Gamma_1^*(x,\xi,t-t_0)\right\}\right]\mathbf{b}_1, & \text{if } t>t_0\end{cases}
$$

where $\alpha = I_0/t_0$ is the ramp rate.

APPENDIX B – 3-STRAND CABLE EXCITED AS IN THE KREMPASKY-SCHMIDT CASE

Consider a 3-strand cable with the conditions:

1. $I(t) = 0$;
2. $r = 0$,
3. even number of periods,
4. $\mathbf{i}^{(0)}(x) = 0$,
5. time-independent excitation spatially bounded to an interval in the center of the cable.

Hypotheses 3 and 5 lead to write:

$$
\mathbf{v}(\xi)=\frac{\dot{\Phi}/2}{\delta}U(\xi-\frac{L}{2}-\frac{\delta}{2})U(\frac{L}{2}+\frac{\delta}{2}-\xi)\begin{Bmatrix}0\\+1\\-1\end{Bmatrix}=\frac{\dot{\Phi}/\sqrt{2}}{\delta}U(\xi-\frac{L}{2}-\frac{\delta}{2})U(\frac{L}{2}+\frac{\delta}{2}-\xi)\mathbf{b}_{-1}=\nu_{-1}(\xi)\mathbf{b}_{-1}
$$

Consequently, the currents flowing in the 3-strand cable are described by the following system:

$$
(P)\ \begin{cases}[G][M]\dfrac{\partial \mathbf{i}}{\partial t}(x,t)+\dfrac{\partial^2 \mathbf{i}}{\partial x^2}(x,t)=[G]\mathbf{v}(x)\\ \mathbf{i}(x,t=0)=\mathbf{i}(x=0,t)=\mathbf{i}(x=L,t)=0\end{cases}
$$

Decomposing **i** as follows

$$
\mathbf{i}(x,t)=\eta_0(x,t)\mathbf{b}_0+\eta_1(x,t)\mathbf{b}_1+\eta_{-1}(x,t)\mathbf{b}_{-1}
$$

Problem (P) is simply divided in the following three problems:

$$
(P_0)\ \begin{cases}\dfrac{\partial^2\eta_0}{\partial x^2}(x,t)=0\\ \eta_0(x,t=0)=\eta_0(x=0,t)=\eta_0(x=L,t)=0\end{cases}
$$

$$
(P_1)\ \begin{cases}\gamma_1\lambda_1\dfrac{\partial\eta_1}{\partial t}(x,t)+\dfrac{\partial^2\eta_1}{\partial x^2}(x,t)=0\\ \eta_1(x,t=0)=\eta_1(x=0,t)=\eta_1(x=L,t)=0\end{cases}
$$

$$(P_{-1})\begin{cases}\gamma_{-1}\lambda_{-1}\dfrac{\partial\eta_{-1}}{\partial t}(x,t)+\dfrac{\partial^2\eta_{-1}}{\partial x^2}(x,t)=\gamma_{-1}\nu_{-1}(x)\\ \eta_{-1}(x,t=0)=\eta_{-1}(x=0,t)=\eta_{-1}(x=L,t)=0\end{cases}$$

Problems (P_0) and (P_1) have the simple solutions

$$\eta_0(x,t)=0 \qquad\qquad \eta_1(x,t)=0$$

While problem (P_{-1}) can be solved as usual. Note that the currents becomes

$$\mathbf{i}(x,t)=\eta_{-1}(x,t)\mathbf{b}_{-1} \quad\Rightarrow\quad i_1(x,t)=0,\; i_2(x,t)=-i_3(x,t)$$

Thus, since

$$\eta_{-1}(x,t)=\frac{2}{L}\int_0^L d\xi\,\frac{\nu_{-1}(\xi)}{\lambda_{-1}}\Gamma_1^*(x,\xi,t)$$

The current i_2 is given by

$$i_2(x,t)=\frac{2}{L}\int_0^L d\xi\,\frac{\nu_{-1}(\xi)}{\lambda_{-1}\sqrt{2}}\Gamma_{-1}^*(x,\xi,t)=\frac{\dot{\Phi}}{L\delta\lambda_{-1}}\int_{(L-\delta)/2}^{(L+\delta)/2}d\xi\,\Gamma_{-1}^*(x,\xi,t)=\frac{\dot{\Phi}}{\delta\lambda_{-1}}\Gamma_{-1}^{**}\left(x,t;\frac{L-\delta}{2},\frac{L+\delta}{2}\right)$$

By using the same notation of Krempasky-Schmidt (but note that the numerical values are different since the eigenvalues for the 3-strand case are different from the 2-strand case) he same solution can be obtained:

$$\Gamma_{-1}^*(x,\xi;t)=\sum_{n=1}^{\infty}\frac{\lambda_{-1}}{\left[-\dfrac{1}{\gamma_{-1}}\left(\dfrac{n\pi}{L}\right)^2\right]}\left\{1-e^{\frac{-t}{\lambda_{-1}}\left[-\frac{1}{\gamma_{-1}}\left(\frac{n\pi}{L}\right)^2\right]}\right\}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)=$$

$$=\frac{3g_{12}\lambda_1 L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)$$

and finally

$$i_2(x,t)=\frac{6}{\pi\alpha}I_m\sum_{\substack{n=1\\ \text{odd } n}}^{\infty}\frac{1}{n^2}\left\{1-e^{-tn^2/\tau}\right\}\sin\left(\frac{n\alpha x}{w}\right)\sin(n\alpha)$$