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AN ANALYTICAL SOLUTION FOR THE CURRENT DISTRIBUTION IN TWO 3-STRANDS RUTHERFORD CABLES COUPLED WITH A RESISTIVE JOINT

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Abstract – **Utilizing the geometrical properties of the auto/mutual induction coefficient matrix among the strands of a 3-strand Rutherford cable, the solution for the general linear case of two cables connected through a resistive joint is given.**

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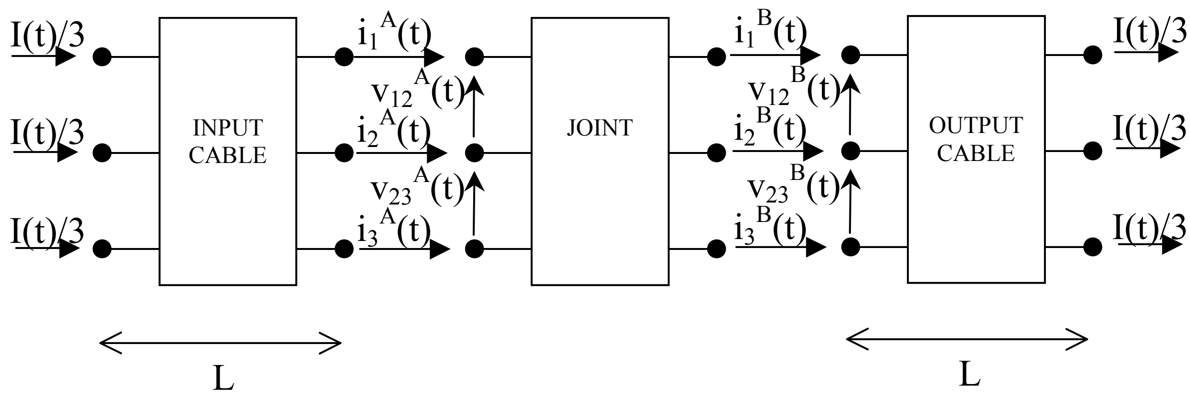
1. Introduction

Consider, as shown in the figure, two equal 3-strand cables of length L connected through a resistive joint. Moreover, assume that the flowing current is equally distributed on the three input strand and on the three output strands.



The longitudinal resistance (r) of the strands ant the external voltage ($\mathbf{v}^{ext}$) applied to the cables are assumed to be zero. The currents and the voltages on the terminals between the input cable and the joint and between the joint and the output cable are labeled with A and B, respectively.

2. Cable Model

Consider, as shown in the figure, the input 3-strand cable with the flowing current equally distributed on the input strands and formally known on the output strands.

The longitudinal resistance (r) and the external voltage (**v**) are assumed to be zero. Consequently, the currents flowing in the 3-strand cable are described by the following system:

$$
\text{(P)}\begin{cases}
[G][M]\dfrac{\partial \mathbf{i}}{\partial t}(x,t)+\dfrac{\partial^2 \mathbf{i}}{\partial x^2}(x,t)=0\\
\mathbf{i}(x,t=0)=0\\
i_1(x=0,t)=i_2(x=0,t)=i_3(x=0,t)=\dfrac{I(t)}{3}\\
\mathbf{i}(x=L,t)=\mathbf{i}^A(t)
\end{cases}
\qquad \text{with } \mathbf{i}, \mathbf{i}^A \in \mathsf{R}^3,\ t>0,\ 0<x<L
$$

with $I(0) = 0$ and $\mathbf{i}^A(0) = 0$, and where [M] and [G] are the following constant-coefficient circulant symmetric matrixes:

$$
[M]=\begin{bmatrix} m_{11} & m_{12} & m_{12}\\ m_{12} & m_{11} & m_{12}\\ m_{12} & m_{12} & m_{11}\end{bmatrix}
\qquad
[G]=\begin{bmatrix} -2g_{12} & g_{12} & g_{12}\\ g_{12} & -2g_{12} & g_{12}\\ g_{12} & g_{12} & -2g_{12}\end{bmatrix}
$$

Defining the orthonormal spectral basis $\mathbf{b}_k$, with $k = 1, 0, -1$, that it's the same for [M] and [G]:

$$
\mathbf{b}_0=\frac{1}{\sqrt{3}}\begin{Bmatrix}1\\1\\1\end{Bmatrix}
\qquad
\mathbf{b}_1=\sqrt{\frac{2}{3}}\begin{Bmatrix}1\\-1/2\\-1/2\end{Bmatrix}
\qquad
\mathbf{b}_{-1}=\frac{1}{\sqrt{2}}\begin{Bmatrix}0\\1\\-1\end{Bmatrix}
$$

the matrixes [M] and [G] can be written as

$$
[M]=\lambda_0\mathbf{b}_0\mathbf{b}_0^T+\lambda_1\mathbf{b}_1\mathbf{b}_1^T+\lambda_{-1}\mathbf{b}_{-1}\mathbf{b}_{-1}^T
\qquad
[G]=\gamma_0\mathbf{b}_0\mathbf{b}_0^T+\gamma_1\mathbf{b}_1\mathbf{b}_1^T+\gamma_{-1}\mathbf{b}_{-1}\mathbf{b}_{-1}^T
$$

with eigenvalues given by

$$
\lambda_0=m_{11}+2m_{12} \qquad \gamma_0=0
$$

$$
\lambda_1=\lambda_{-1}=m_{11}-m_{12} \qquad \gamma_1=\gamma_{-1}=-3g_{12}
$$

§ 2.1 - Time domain Currents

Decompose $\mathbf{i}$ and $\mathbf{i}^A$ as follows

$$
\mathbf{i}(x,t)=\eta_0(x,t)\mathbf{b}_0+\eta_1(x,t)\mathbf{b}_1+\eta_{-1}(x,t)\mathbf{b}_{-1}
$$

$$
\mathbf{i}^A(t)=\eta_0^A(t)\mathbf{b}_0+\eta_1^A(t)\mathbf{b}_1+\eta_{-1}^A(t)\mathbf{b}_{-1}
$$

Note that (compatibility condition)

$$
\eta_0^A(t)=\mathbf{b}_0^T\mathbf{i}^A(t)=\frac{1}{\sqrt{3}}\left[i_1^A(t)+i_2^A(t)+i_3^A(t)\right]=\frac{I(t)}{\sqrt{3}}
$$

Problem (P) is stated as:

$$
\text{(P)}\begin{cases}
[G][M]\dfrac{\partial \mathbf{i}}{\partial t}(x,t)+\dfrac{\partial^2 \mathbf{i}}{\partial x^2}(x,t)=0\\
\mathbf{i}(x,t=0)=0\\
\mathbf{i}(x=0,t)=\dfrac{I(t)}{\sqrt{3}}\mathbf{b}_0\\
\mathbf{i}(x=L,t)=\mathbf{i}^A(t)
\end{cases}
\qquad \text{with } t>0,\ 0<x<L
$$

Defining

$$\mathbf{u}(x,t)=\mathbf{i}(x,t)-\frac{I(t)}{\sqrt{3}}\left(1-\frac{x}{L}\right)\mathbf{b}_0-\mathbf{i}^A(t)\frac{x}{L}$$

gives

$$(P_u)\begin{cases}[G][M]\dfrac{\partial \mathbf{u}}{\partial t}(x,t)+\dfrac{\partial^2 \mathbf{u}}{\partial x^2}(x,t)=-[G][M]\left(\dfrac{I'(t)}{\sqrt{3}}\left(1-\dfrac{x}{L}\right)\mathbf{b}_0+\mathbf{i}'^A(t)\dfrac{x}{L}\right)\\ \mathbf{u}(x,t=0)=0\\ \mathbf{u}(x=0,t)=0\\ \mathbf{u}(x=L,t)=0\end{cases}\quad \text{with } t>0,\ 0<x<L$$

where the prime denotes differentiation with respect to the argument. Decomposing **u** as follows

$$\mathbf{u}(x,t)=\mu_0(x,t)\mathbf{b}_0+\mu_1(x,t)\mathbf{b}_1+\mu_{-1}(x,t)\mathbf{b}_{-1}$$

Problem (P_u) is simply divided in the following three problems:

$$(P_{u0})\begin{cases}\dfrac{\partial^2\mu_0}{\partial x^2}(x,t)=0\\ \mu_0(x,t=0)=0,\quad \mu_0(x=0,t)=\mu_0(x=L,t)=0\end{cases}\quad \text{with } t>0,\ 0<x<L$$

$$(P_{u1})\begin{cases}\gamma_1\lambda_1\dfrac{\partial\mu_1}{\partial t}(x,t)+\dfrac{\partial^2\mu_1}{\partial x^2}(x,t)=-\gamma_1\lambda_1\eta_1'^A(t)\dfrac{x}{L}\\ \mu_1(x,t=0)=0,\quad \mu_1(x=0,t)=\mu_1(x=L,t)=0\end{cases}\quad \text{with } t>0,\ 0<x<L$$

$$(P_{u-1})\begin{cases}\gamma_{-1}\lambda_{-1}\dfrac{\partial\mu_{-1}}{\partial t}(x,t)+\dfrac{\partial^2\mu_{-1}}{\partial x^2}(x,t)=-\gamma_{-1}\lambda_{-1}\eta_{-1}'^A(t)\dfrac{x}{L}\\ \mu_{-1}(x,t=0)=0,\quad \mu_{-1}(x=0,t)=\mu_{-1}(x=L,t)=0\end{cases}\quad \text{with } t>0,\ 0<x<L$$

Problem (P_{u0}) has the simple solution $\mu_0(x,t)=0$, while problems ($P_{u\pm1}$) can be written compactly as:

$$\begin{cases}\gamma_{\pm1}\lambda_{\pm1}\dfrac{\partial\mu_{\pm1}}{\partial t}(x,t)+\dfrac{\partial^2\mu_{\pm1}}{\partial x^2}(x,t)=-\gamma_{\pm1}\lambda_{\pm1}\eta_{\pm1}'^A(t)\dfrac{x}{L}\\ \mu_{\pm1}(x,t=0)=0,\quad \mu_{\pm1}(x=0,t)=\mu_{\pm1}(x=L,t)=0\end{cases}\quad \text{with } t>0,\ 0<x<L$$

which has been already treated [1] and gives:

$$\mu_{\pm1}(x,t)=-\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\ \eta_{\pm1}'^A(t-\tau)\frac{\xi}{L}\Gamma(x,\xi,\tau)$$

where Γ has the two equivalent representations [1]:

$$\Gamma(x,\xi;t)=\sum_{n=1}^{\infty} e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)$$

or

$$\Gamma(x,\xi;t) = \frac{L}{4}\sqrt{-\frac{\lambda\gamma}{\pi t}}\sum_{n=-\infty}^{\infty}\left[e^{\frac{\lambda\gamma}{t}\left(\frac{x-\xi}{2}-nL\right)^2} - e^{\frac{\lambda\gamma}{t}\left(\frac{x+\xi}{2}-nL\right)^2}\right]$$

(The indexes has been dropped from the eigenvalues since $\lambda_1 = \lambda_{-1}$ and $\gamma_1 = \gamma_{-1}$). Thus since

$$\mu_0(x,t) = \eta_0(x,t) - \frac{I(t)}{\sqrt{3}}\left(1-\frac{x}{L}\right) - \mathbf{b}_0^T\mathbf{i}^A(t)\frac{x}{L} = \eta_0(x,t) - \frac{I(t)}{\sqrt{3}} + \frac{I(t)}{\sqrt{3}}\frac{x}{L} - \frac{I(t)}{\sqrt{3}}\frac{x}{L} = \eta_0(x,t) - \frac{I(t)}{\sqrt{3}}$$

$$\mu_{\pm1}(x,t) = \eta_{\pm1}(x,t) - \mathbf{b}_{\pm1}^T\mathbf{i}^A(t)\frac{x}{L} = \eta_{\pm1}(x,t) - \eta_{\pm1}^A(t)\frac{x}{L}$$

which implies

$$\eta_0(x,t) = \frac{I(t)}{\sqrt{3}}$$

$$\eta_{\pm1}(x,t) = \eta_{\pm1}^A(t)\frac{x}{L} - \frac{2}{L}\int_0^L d\xi\int_0^t d\tau\, \eta'^A_{\pm1}(t-\tau)\frac{\xi}{L}\Gamma(x,\xi,\tau) \quad \text{with } t>0,\ 0<x<L$$

and the <u>currents in the inlet cable</u> are (for $t > 0$, $0 < x < L$):

$$\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{3}}\mathbf{b}_0 + \left[\eta_1^A(t)\mathbf{b}_1 + \eta_{-1}^A(t)\mathbf{b}_{-1}\right]\frac{x}{L} - \frac{2}{L}\int_0^L d\xi\int_0^t d\tau\left[\eta_1'^A(t-\tau)\mathbf{b}_1 + \eta_{-1}'^A(t-\tau)\mathbf{b}_{-1}\right]\frac{\xi}{L}\Gamma(x,\xi,\tau)$$

Note that the currents in the inlet cable are completely known if the functions $\eta_{\pm1}{}^A$ are available.

For what concerns the <u>currents in the outlet cable</u>, since it's formally symmetric to the inlet one, the solution for **i**, and for $\eta_{\pm1}$ also, can be obtained changing A with B and considering $2L - x$ instead of x, as follows(*):

$$\eta_0(x,t) = \frac{I(t)}{\sqrt{3}}$$

$$\eta_{\pm1}(x,t) = \eta_{\pm1}^B(t)\frac{2L-x}{L} - \frac{2}{L}\int_0^L d\xi\int_0^t d\tau\, \eta'^B_{\pm1}(t-\tau)\frac{\xi}{L}\Gamma(2L-x,\xi,\tau) \qquad \text{with } t>0,\ L<x<2L$$

and the currents in the outlet cable are (for $t > 0$, $L < x < 2L$):

$$\mathbf{i}(x,t) = \frac{I(t)}{\sqrt{3}}\mathbf{b}_0 + \left[\eta_1^B(t)\mathbf{b}_1 + \eta_{-1}^B(t)\mathbf{b}_{-1}\right]\frac{2L-x}{L} - \frac{2}{L}\int_0^L d\xi\int_0^t d\tau\left[\eta_1'^B(t-\tau)\mathbf{b}_1 + \eta_{-1}'^B(t-\tau)\mathbf{b}_{-1}\right]\frac{\xi}{L}\Gamma(2L-x,\xi,\tau)$$

Again, note that the currents in the outlet cable are completely known if the functions $\eta_{\pm1}{}^B$ are available.

§ 2.2 - Time domain Voltages

(*) The correspondence is simply demonstrated, since defining x' = 2L –x, the points x = L becomes x' = L, the points x = 2L becomes x' = 0, and the derivatives results to be: $\frac{\partial}{\partial x} = -\frac{\partial}{\partial x'}$ and $\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x'^2}$. Thus the leading equation is invariant and the boundary conditions correctly match.

For what concerns the voltages in the inlet cable, taking into account that

$$[G]\mathbf{v}(x,t) = \frac{\partial \mathbf{i}}{\partial x}(x,t) \qquad \text{with } t > 0,\ 0 < x < L$$

and decomposing **v** as follows

$$\mathbf{v}(x,t) = \psi_0(x,t)\mathbf{b}_0 + \psi_1(x,t)\mathbf{b}_1 + \psi_{-1}(x,t)\mathbf{b}_{-1}$$

the following relations are obtained:

$$\gamma_{\pm1}\psi_{\pm1}(x,t) = \frac{\partial \eta_{\pm1}}{\partial x}(x,t) \qquad \text{with } t > 0,\ 0 < x < L$$

Moreover, since

$$\psi_{-1}(x,t) = \mathbf{b}_{-1}^{T}\mathbf{v}(x,t) = \frac{1}{\sqrt{2}}[v_2(x,t) - v_3(x,t)] = \frac{v_{23}(x,t)}{\sqrt{2}}$$

$$\psi_1(x,t) = \mathbf{b}_1^{T}\mathbf{v}(x,t) = \sqrt{\frac{2}{3}}\left[v_1(x,t) - \frac{v_2(x,t)}{2} - \frac{v_3(x,t)}{2}\right] = \sqrt{\frac{2}{3}}v_{12}(x,t) + \frac{1}{\sqrt{6}}v_{23}(x,t)$$

and $v_{12}^{A}(t) = v_{12}(x = L, t)$, $v_{23}^{A}(t) = v_{23}(x = L, t)$ the voltages at the end of the inlet cable can be stated as:

$$v_{23}^{A}(t) = \sqrt{2}\,\psi_{-1}(x = L,t) \qquad v_{12}^{A}(t) = \sqrt{\frac{3}{2}}\,\psi_1(x = L,t) - \frac{1}{\sqrt{2}}\psi_{-1}(x = L,t)$$

Now, since the voltages $v_{12}^{A}(t)$ and $v_{23}^{A}(t)$ at the end of the inlet cable are given in terms of $\psi_{\pm1}(x, t)$, that are related to the spatial derivatives of $\eta_{\pm1}(x, t)$ which are dependent from the currents at the end of the inlet cable (expressed in terms of $\eta_{\pm1}^{A}(t)$), it's apparent that it is possible to obtain a voltage- current characteristic of the inlet cable, that can be seen as an active tripolar component.

For what concerns the voltages in the outlet cable, since it's formally symmetric to the inlet one, the solution for $v_{12}^{B}(t)$ and $v_{23}^{B}(t)$, can be obtained changing A with B and considering 2L – x instead of x, as follows:

$$v_{23}^{B}(t) = \sqrt{2}\,\psi_{-1}(x = L,t) \qquad v_{12}^{B}(t) = \sqrt{\frac{3}{2}}\,\psi_1(x = L,t) - \frac{1}{\sqrt{2}}\psi_{-1}(x = L,t)$$

(where $\psi_{\pm1}$ have to be evaluated referring to the spatial derivatives of $\eta_{\pm1}$ evaluated for the outlet cable)

To obtain the voltage-current characteristic for the inlet cable, define:

$$\underline{\Delta}(x,t) = 1 - L\frac{\partial}{\partial x}\left[\frac{2}{L}\int_0^L d\xi\,\frac{\xi}{L}\,\Gamma(x,\xi,t)\right] \qquad \text{with } t > 0,\ 0 < x < L$$

Thus, substitution gives:

$$\gamma_{\pm1}\psi_{\pm1}(x,t)=\frac{\partial\eta_{\pm1}}{\partial x}(x,t)=\eta^{A}_{\pm1}(t)\frac{1}{L}-\frac{\partial}{\partial x}\frac{2}{L}\int_0^L d\xi\int_0^t d\tau\ \eta'^{A}_{\pm1}(t-\tau)\frac{\xi}{L}\Gamma(x,\xi,\tau)=$$

$$=\eta^{A}_{\pm1}(t)\frac{1}{L}+\int_0^t d\tau\ \eta'^{A}_{\pm1}(t-\tau)\frac{\underline{\Delta}(x,\tau)-1}{L}= \qquad t>0,\ 0<x<L$$

$$=\eta^{A}_{\pm1}(t)\frac{1}{L}+\frac{1}{L}\int_0^t d\tau\ \eta'^{A}_{\pm1}(t-\tau)\underline{\Delta}(x,\tau)-\frac{1}{L}\left[\eta^{A}_{\pm1}(\tau)\right]_{\tau=0}^{\tau=t}$$

and, since $\eta_{\pm1}{}^{A}(t=0)=0$, this gives:

$$\psi_{\pm1}(x,t)=\frac{1}{\gamma_{\pm1}L}\int_0^t d\tau\ \eta'^{A}_{\pm1}(t-\tau)\underline{\Delta}(x,\tau) \qquad \text{with } t>0,\ 0<x<L$$

Moreover, in the same way the voltage-current characteristic for the outlet cable, are obtained:

$$\psi_{\pm1}(x,t)=-\frac{1}{\gamma_{\pm1}L}\int_0^t d\tau\ \eta'^{B}_{\pm1}(t-\tau)\underline{\Delta}(2L-x,\tau) \qquad \text{with } t>0,\ L<x<2L$$

(the sign is due to the x-differentiation: see footnote at p. 4) Finally, the voltages at the end of the inlet cable can be stated as:

$$v^{A}_{23}(t)=\frac{\sqrt{2}}{\gamma L}\int_0^t d\tau\ \eta'^{A}_{-1}(t-\tau)\underline{\Delta}(L,\tau)$$

$$v^{A}_{12}(t)=\frac{1}{\gamma L\sqrt{2}}\int_0^t d\tau\ \left[\sqrt{3}\eta'^{A}_{+1}(t-\tau)-\eta'^{A}_{-1}(t-\tau)\right]\underline{\Delta}(L,\tau)$$

(where the index on γ has been dropped for brevity) and the voltages at the beginning of the outlet cable results to be:

$$v^{B}_{23}(t)=-\frac{\sqrt{2}}{\gamma L}\int_0^t d\tau\ \eta'^{B}_{-1}(t-\tau)\underline{\Delta}(L,\tau)$$

$$v^{B}_{12}(t)=-\frac{1}{\gamma L\sqrt{2}}\int_0^t d\tau\ \left[\sqrt{3}\eta'^{B}_{+1}(t-\tau)-\eta'^{B}_{-1}(t-\tau)\right]\underline{\Delta}(L,\tau)$$

§ 2.3 - Laplace domain Currents

As it was shown, the currents flowing in the inlet 3-strand cable are described by the following problem (P):

$$(P)\begin{cases}[G][M]\dfrac{\partial\mathbf{i}}{\partial t}(x,t)+\dfrac{\partial^2\mathbf{i}}{\partial x^2}(x,t)=0\\ \mathbf{i}(x,t=0)=0\\ \mathbf{i}(x=0,t)=\dfrac{I(t)}{\sqrt{3}}\mathbf{b}_0\\ \mathbf{i}(x=L,t)=\mathbf{i}^{A}(t)\end{cases} \qquad \text{with } t>0,\ 0<x<L$$

Decompose $\mathbf{i}$ and $\mathbf{i}^A$ as follows

$$\mathbf{i}(x,t) = \eta_0(x,t)\mathbf{b}_0 + \eta_1(x,t)\mathbf{b}_1 + \eta_{-1}(x,t)\mathbf{b}_{-1}$$

$$\mathbf{i}^A(t) = \eta_0^A(t)\mathbf{b}_0 + \eta_1^A(t)\mathbf{b}_1 + \eta_{-1}^A(t)\mathbf{b}_{-1}$$

with the compatibility condition

$$\eta_0^A(t) = \mathbf{b}_0^T\mathbf{i}^A(t) = \frac{1}{\sqrt{3}}\left[i_1^A(t) + i_2^A(t) + i_3^A(t)\right] = \frac{I(t)}{\sqrt{3}}$$

Problem (P) is simply divided in the following three problems:

$$(P_0)\ \begin{cases} \dfrac{\partial^2\eta_0}{\partial x^2}(x,t) = 0 \\ \eta_0(x,t=0) = 0, \quad \eta_0(x=0,t) = \eta_0(x=L,t) = \dfrac{I(t)}{\sqrt{3}} \end{cases} \qquad \text{with } t > 0,\ 0 < x < L$$

$$(P_1)\ \begin{cases} \gamma_1\lambda_1\dfrac{\partial\eta_1}{\partial t}(x,t) + \dfrac{\partial^2\eta_1}{\partial x^2}(x,t) = 0 \\ \eta_1(x,t=0) = 0, \quad \eta_1(x=0,t) = 0, \quad \eta_1(x=L,t) = \eta_1^A(t) \end{cases} \qquad \text{with } t > 0,\ 0 < x < L$$

$$(P_{-1})\ \begin{cases} \gamma_{-1}\lambda_{-1}\dfrac{\partial\eta_{-1}}{\partial t}(x,t) + \dfrac{\partial^2\eta_{-1}}{\partial x^2}(x,t) = 0 \\ \eta_{-1}(x,t=0) = 0, \quad \eta_{-1}(x=0,t) = 0, \quad \eta_{-1}(x=L,t) = \eta_{-1}^A(t) \end{cases} \qquad \text{with } t > 0,\ 0 < x < L$$

Problem (P_0) has the simple solution $\eta_0(x,t) = \dfrac{I(t)}{\sqrt{3}}$, while problems ($P_{\pm1}$) can be written compactly as:

$$(P_{\pm1})\ \begin{cases} \gamma_{\pm1}\lambda_{\pm1}\dfrac{\partial\eta_{\pm1}}{\partial t}(x,t) + \dfrac{\partial^2\eta_{\pm1}}{\partial x^2}(x,t) = 0 \\ \eta_{\pm1}(x,t=0) = 0, \quad \eta_{\pm1}(x=0,t) = 0, \quad \eta_{\pm1}(x=L,t) = \eta_{\pm1}^A(t) \end{cases} \qquad \text{with } t > 0,\ 0 < x < L$$

Applying the Laplace transform L to problems ($P_{\pm1}$) gives:

$$(\mathsf{L}P_{\pm1})\ \begin{cases} s\gamma_{\pm1}\lambda_{\pm1}\tilde{\eta}_{\pm1}(x,s) + \dfrac{\partial^2\tilde{\eta}_{\pm1}}{\partial x^2}(x,s) = 0 \\ \tilde{\eta}_{\pm1}(x=0,s) = 0, \quad \tilde{\eta}_{\pm1}(x=L,s) = \tilde{\eta}_{\pm1}^A(s) \end{cases} \qquad \text{with } 0 < x < L$$

which can be easily solved to give: $\tilde{\eta}_{\pm1}(x,s) = \tilde{\eta}_{\pm1}^A(s)\dfrac{\sin\left(x\sqrt{s\gamma_{\pm1}\lambda_{\pm1}}\right)}{\sin\left(L\sqrt{s\gamma_{\pm1}\lambda_{\pm1}}\right)}$

Consider now the solution for $\eta_{\pm1}(x, t)$ given before:

$$\eta_{\pm1}(x,t) = \eta_{\pm1}^A(t)\frac{x}{L} - \frac{2}{L}\int_0^L d\xi\int_0^t d\tau\ \eta_{\pm1}'^A(t-\tau)\frac{\xi}{L}\Gamma(x,\xi,\tau) \qquad \text{with } t > 0,\ 0 < x < L$$

Noting that the time integration is of the convolution kind it can be easily Laplace-transformed:

$$\tilde{\eta}_{\pm1}(x,s)=\tilde{\eta}^{A}_{\pm1}(s)\frac{x}{L}-s\tilde{\eta}^{A}_{\pm1}(s)\ \mathsf{L}_s\left\{\frac{2}{L}\int_0^L d\xi\ \frac{\xi}{L}\Gamma(x,\xi,t)\right\}=\tilde{\eta}^{A}_{\pm1}(s)\left[\frac{x}{L}-s\ \mathsf{L}_s\left\{\frac{2}{L}\int_0^L d\xi\ \frac{\xi}{L}\Gamma(x,\xi,t)\right\}\right]$$

By comparison it can be deduced that

$$\frac{x}{L}-s\ \mathsf{L}_s\left\{\frac{2}{L}\int_0^L d\xi\ \frac{\xi}{L}\Gamma(x,\xi,t)\right\}=\frac{\sin\left(x\sqrt{s\gamma\lambda}\right)}{\sin\left(L\sqrt{s\gamma\lambda}\right)}$$

where the index has been dropped for brevity. To obtain the Laplace-transform of the characteristic function $\underline{\Delta}$, differentiate the last equation with respect to x:

$$\frac{1}{L}-s\ \mathsf{L}_s\left\{\frac{\partial}{\partial x}\frac{2}{L}\int_0^L d\xi\ \frac{\xi}{L}\Gamma(x,\xi,t)\right\}=\sqrt{s\gamma\lambda}\,\frac{\cos\left(x\sqrt{s\gamma\lambda}\right)}{\sin\left(L\sqrt{s\gamma\lambda}\right)}$$

Taking into account the definition

$$\underline{\Delta}(x,t)=1-L\frac{\partial}{\partial x}\left[\frac{2}{L}\int_0^L d\xi\frac{\xi}{L}\Gamma(x,\xi,t)\right]$$

and since $\mathsf{L}_s\{1\}=1/s$, it results:

$$\sqrt{s\gamma\lambda}\,\frac{\cos\left(x\sqrt{s\gamma\lambda}\right)}{\sin\left(L\sqrt{s\gamma\lambda}\right)}=\frac{1}{L}-s\ \mathsf{L}_s\left\{\frac{1}{L}-\frac{\underline{\Delta}(x,t)}{L}\right\}=\frac{s}{L}\tilde{\underline{\Delta}}(x,s)$$

where $\tilde{\underline{\Delta}}(x,s)=\ \mathsf{L}_s\{\underline{\Delta}(x,t)\}$. Thus, finally

$s\tilde{\underline{\Delta}}(x,s)=L\sqrt{s\gamma\lambda}\,\frac{\cos\left(x\sqrt{s\gamma\lambda}\right)}{\sin\left(L\sqrt{s\gamma\lambda}\right)}$	$\tilde{\underline{\Delta}}(L,s)=\frac{L}{s}\sqrt{s\gamma\lambda}\,\frac{\cos\left(L\sqrt{s\gamma\lambda}\right)}{\sin\left(L\sqrt{s\gamma\lambda}\right)}$

§ 2.4 - Laplace domain Voltages

As it was shown before, the voltages at the end of the inlet cable and at the beginning of the outlet cable can be stated as time-convolutions. Therefore, they can be easily transformed in the Laplace domain:

$\tilde{v}^{A}_{23}(s)=\frac{\sqrt{2}}{\gamma L}\tilde{\eta}^{A}_{-1}(s)s\tilde{\underline{\Delta}}(L,s)$	$\tilde{v}^{B}_{23}(s)=-\frac{\sqrt{2}}{\gamma L}\tilde{\eta}^{B}_{-1}(s)s\tilde{\underline{\Delta}}(L,s)$
$\tilde{v}^{A}_{12}(s)=\frac{1}{\gamma L\sqrt{2}}\left[\sqrt{3}\tilde{\eta}^{A}_{+1}(s)-\tilde{\eta}^{A}_{-1}(s)\right]s\tilde{\underline{\Delta}}(L,s)$	$\tilde{v}^{B}_{12}(s)=-\frac{1}{\gamma L\sqrt{2}}\left[\sqrt{3}\tilde{\eta}^{B}_{+1}(s)-\tilde{\eta}^{B}_{-1}(s)\right]s\tilde{\underline{\Delta}}(L,s)$

§ 2.5 – Properties of the $\underline{\Delta}$ function

As it was shown before, the current-voltage characteristic of the inlet and outlet cables are summarized in the $\underline{\Delta}$ function. In this section some properties of the $\underline{\Delta}$ function are given. Starting from the definitions:

$$\Gamma(x,\xi;t)=\sum_{n=1}^{\infty}e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}\ \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi\xi}{L}\right)\qquad\qquad \underline{\Delta}(x,t)=1-L\frac{\partial}{\partial x}\left[\frac{2}{L}\int_0^L d\xi\frac{\xi}{L}\Gamma(x,\xi,t)\right]$$

($0<\xi<L$ and $0<x<L$) and taking into account that

$$\frac{1}{L}\int_0^L d\xi\,\frac{\xi}{L}\sin\left(\frac{n\pi\xi}{L}\right)=\frac{1}{(n\pi)^2}\left[\sin\left(\frac{n\pi\xi}{L}\right)-\left(\frac{n\pi\xi}{L}\right)\cos\left(\frac{n\pi\xi}{L}\right)\right]_0^L=-\frac{\cos(n\pi)}{n\pi}$$

gives

$$\underline{\Delta}(x,t)=1+2\sum_{n=1}^{\infty} e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}\cos\left(\frac{n\pi x}{L}\right)\cos(n\pi)=\sum_{n=-\infty}^{+\infty} e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}\cos\left(\frac{n\pi x}{L}\right)\cos(n\pi) \qquad ,\ 0<x<L$$

In order to obtain a different (and with faster convergence) representation of the $\underline{\Delta}$ function consider that $\cos(p)\cos(q)=\frac{1}{2}\left[\cos(p-q)+\cos(p+q)\right]$ and that the elliptic theta function ϑ_3 admit the following representation (where the summation terms are non-oscillating)[2]:

$$\vartheta_3\left(u,e^{-s}\right)=\sum_{n=-\infty}^{\infty} e^{-n^2 s}\cos 2nu=\sqrt{\frac{\pi}{s}}\sum_{n=-\infty}^{\infty} e^{-\frac{(u-n\pi)^2}{s}}$$

Therefore

$$\underline{\Delta}(x,t)=\frac{1}{2}\sum_{n=-\infty}^{+\infty} e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}\cos\left(n\pi\left(1+\frac{x}{L}\right)\right)+\frac{1}{2}\sum_{n=-\infty}^{+\infty} e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}\cos\left(n\pi\left(1-\frac{x}{L}\right)\right)=$$

$$=\frac{1}{2}\vartheta_3\left(\frac{\pi}{2}\left(1+\frac{x}{L}\right),e^{\frac{t}{\lambda\gamma}\left(\frac{\pi}{L}\right)^2}\right)+\frac{1}{2}\vartheta_3\left(\frac{\pi}{2}\left(1-\frac{x}{L}\right),e^{\frac{t}{\lambda\gamma}\left(\frac{\pi}{L}\right)^2}\right)=$$

$$=\frac{L}{2}\sqrt{\frac{-\lambda\gamma}{\pi t}}\sum_{n=-\infty}^{+\infty} e^{\frac{\lambda\gamma}{t}\left(\frac{L+x}{2}-nL\right)^2}+\frac{L}{2}\sqrt{\frac{-\lambda\gamma}{\pi t}}\sum_{n=-\infty}^{+\infty} e^{\frac{\lambda\gamma}{t}\left(\frac{L-x}{2}-nL\right)^2}$$

that gives

$$\underline{\Delta}(x,t)=\frac{L}{2}\sqrt{\frac{-\lambda\gamma}{\pi t}}\sum_{n=-\infty}^{+\infty}\left[e^{\frac{\lambda\gamma}{t}\left(\frac{L+x}{2}-nL\right)^2}+e^{\frac{\lambda\gamma}{t}\left(\frac{L-x}{2}-nL\right)^2}\right] \qquad ,\ 0<x<L$$

In particular, $\underline{\Delta}(L, t)$ becomes:

$\underline{\Delta}(L,t)=\sum_{n=-\infty}^{+\infty} e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}$	$\underline{\Delta}(L,t)=L\sqrt{\frac{-\lambda\gamma}{\pi t}}\sum_{n=-\infty}^{+\infty} e^{\frac{\lambda\gamma}{t}(nL)^2}$

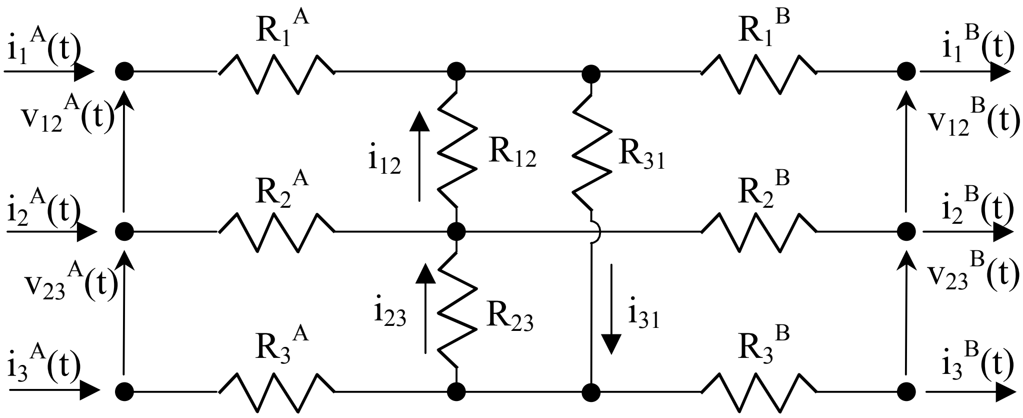
Correspondingly, in the Laplace domain $\tilde{\underline{\Delta}}(L,s)$ has the following three different representations [3]:

$\tilde{\underline{\Delta}}(L,s)=\sum_{n=-\infty}^{+\infty}\frac{1}{s-\frac{1}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}$	$\tilde{\underline{\Delta}}(L,s)=L\sqrt{\frac{-\lambda\gamma}{s}}\sum_{n=-\infty}^{+\infty} e^{-2L\sqrt{-s\lambda\gamma}\lvert n\rvert}$	$\tilde{\underline{\Delta}}(L,s)=L\sqrt{\frac{\gamma\lambda}{s}}\frac{\cos\left(L\sqrt{s\gamma\lambda}\right)}{\sin\left(L\sqrt{s\gamma\lambda}\right)}$

3. Joint Model

§ 3.1 – Time domain Model

Consider, as shown in the figure, the resistive joint with the flowing current known formally on the input and on the output strands.



The LKC on the central nodes gives:

$$\begin{cases} i_1^A(t) + i_{12}(t) = i_1^B(t) + i_{31}(t) \\ i_2^A(t) + i_{23}(t) = i_2^B(t) + i_{12}(t) \\ i_3^A(t) + i_{31}(t) = i_3^B(t) + i_{23}(t) \end{cases}$$

and, defining

$$\mathbf{i}^Q(t) = \begin{Bmatrix} i_{12} \\ i_{23} \\ i_{31} \end{Bmatrix} \qquad [\Omega] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

the LKC can be compactly written as:

$$\mathbf{i}^A(t) + \mathbf{i}^Q(t) = \mathbf{i}^B(t) + [\Omega]\mathbf{i}^Q(t)$$

Now note that

$$[\Omega]\mathbf{b}_0 = \mathbf{b}_0 \qquad [\Omega]\mathbf{b}_1 = -\frac{1}{2}\mathbf{b}_1 + \frac{\sqrt{3}}{2}\mathbf{b}_{-1} \qquad [\Omega]\mathbf{b}_{-1} = -\frac{\sqrt{3}}{2}\mathbf{b}_1 - \frac{1}{2}\mathbf{b}_{-1}$$

Decomposing $\mathbf{i}^A$, $\mathbf{i}^B$ and $\mathbf{i}^Q$ as follows

$$\mathbf{i}^A(t) = \eta_0^A(t)\mathbf{b}_0 + \eta_1^A(t)\mathbf{b}_1 + \eta_{-1}^A(t)\mathbf{b}_{-1}$$

$$\mathbf{i}^B(t) = \eta_0^B(t)\mathbf{b}_0 + \eta_1^B(t)\mathbf{b}_1 + \eta_{-1}^B(t)\mathbf{b}_{-1}$$

$$\mathbf{i}^Q(t) = \eta_0^Q(t)\mathbf{b}_0 + \eta_1^Q(t)\mathbf{b}_1 + \eta_{-1}^Q(t)\mathbf{b}_{-1}$$

and noting that (compatibility condition)

$$\eta_0^A(t) = \eta_0^B(t) = \frac{I(t)}{\sqrt{3}}$$

the following equations holds:

$$\begin{cases} \eta_1^A(t) + \eta_1^Q(t) = \eta_1^B(t) - \frac{1}{2}\eta_1^Q(t) - \frac{\sqrt{3}}{2}\eta_{-1}^Q(t) \\ \eta_{-1}^A(t) + \eta_{-1}^Q(t) = \eta_{-1}^B(t) + \frac{\sqrt{3}}{2}\eta_1^Q(t) - \frac{1}{2}\eta_{-1}^Q(t) \end{cases}$$

This system can be easily solved for $\eta_{\pm 1}{}^Q$, leading to:

$$\begin{cases} \eta_1^Q = \frac{1}{2\sqrt{3}}\left[\sqrt{3}\left(\eta_1^B - \eta_1^A\right) - \left(\eta_{-1}^B - \eta_{-1}^A\right)\right] \\ \eta_{-1}^Q = \frac{1}{2\sqrt{3}}\left[\left(\eta_1^B - \eta_1^A\right) + \sqrt{3}\left(\eta_{-1}^B - \eta_{-1}^A\right)\right] \end{cases}$$

The LKT on the left and right sides of the joint gives:

$$\begin{cases} v_{12}^A(t) = R_1^A i_1^A(t) - R_2^A i_2^A(t) - R_{12} i_{12}(t) \\ v_{23}^A(t) = R_2^A i_2^A(t) - R_3^A i_3^A(t) - R_{23} i_{23}(t) \\ v_{31}^A(t) = R_3^A i_3^A(t) - R_1^A i_1^A(t) - R_{31} i_{31}(t) \end{cases} \qquad \begin{cases} v_{12}^B(t) = -R_1^B i_1^B(t) + R_2^B i_2^B(t) - R_{12} i_{12}(t) \\ v_{23}^B(t) = -R_2^B i_2^B(t) + R_3^B i_3^B(t) - R_{23} i_{23}(t) \\ v_{31}^B(t) = -R_3^B i_3^B(t) + R_1^B i_1^B(t) - R_{31} i_{31}(t) \end{cases}$$

and, defining

$$\mathbf{v}^A(t) = \begin{Bmatrix} v_{12}^A \\ v_{23}^A \\ v_{31}^A \end{Bmatrix} \qquad \mathbf{v}^B(t) = \begin{Bmatrix} v_{12}^B \\ v_{23}^B \\ v_{31}^B \end{Bmatrix} \qquad \left[R^A\right] = \begin{bmatrix} R_1^A & -R_2^A & 0 \\ 0 & R_2^A & -R_3^A \\ -R_1^A & 0 & R_3^A \end{bmatrix} \qquad \left[R^B\right] = \begin{bmatrix} R_1^B & -R_2^B & 0 \\ 0 & R_2^B & -R_3^B \\ -R_1^B & 0 & R_3^B \end{bmatrix}$$

$\left[R^Q\right] = \mathrm{diag}\{R_{12}, R_{23}, R_{31}\}$, the LKT can be compactly written as:

$$\mathbf{v}^A(t) = \left[R^A\right]\mathbf{i}^A(t) - \left[R^Q\right]\mathbf{i}^Q(t) \qquad \mathbf{v}^B(t) = -\left[R^B\right]\mathbf{i}^B(t) - \left[R^Q\right]\mathbf{i}^Q(t)$$

Moreover, note that

$$\mathbf{b}_0^T\left[R^A\right] = \mathbf{b}_0^T\left[R^B\right] = 0 \qquad v_{12}^A(t) + v_{23}^A(t) + v_{31}^A(t) = v_{12}^B(t) + v_{23}^B(t) + v_{31}^B(t) = 0$$

This last property led us to write:

$$\mathbf{v}^A(t) = \begin{Bmatrix} v_{12}^A \\ v_{23}^A \\ -v_{12}^A - v_{23}^A \end{Bmatrix} = v_{12}^A\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} + v_{23}^A\begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} = v_{12}^A\sqrt{\frac{3}{2}}\mathbf{b}_1 + \left(\frac{v_{12}^A}{\sqrt{2}} + \sqrt{2}\,v_{23}^A\right)\mathbf{b}_{-1}$$

(the same is true for $\mathbf{v}^B$) Thus, decomposing $\mathbf{v}^A$ and $\mathbf{v}^B$ as follows

$$\mathbf{v}^A(t) = \beta_0^A(t)\mathbf{b}_0 + \beta_1^A(t)\mathbf{b}_1 + \beta_{-1}^A(t)\mathbf{b}_{-1}$$

$$\mathbf{v}^B(t) = \beta_0^B(t)\mathbf{b}_0 + \beta_1^B(t)\mathbf{b}_1 + \beta_{-1}^B(t)\mathbf{b}_{-1}$$

it can be easily proven then

$$\begin{cases} \beta_0^A = 0 \\ \beta_1^A = v_{12}^A\sqrt{\frac{3}{2}} \\ \beta_{-1}^A = \frac{v_{12}^A}{\sqrt{2}} + \sqrt{2}\,v_{23}^A \end{cases} \qquad \begin{cases} \beta_0^B = 0 \\ \beta_1^B = v_{12}^B\sqrt{\frac{3}{2}} \\ \beta_{-1}^B = \frac{v_{12}^B}{\sqrt{2}} + \sqrt{2}\,v_{23}^B \end{cases}$$

Finally, defining

$$\rho_{h,k}^Z = \mathbf{b}_h^T\left[R^Z\right]\mathbf{b}_k \text{ , with h, k = 0, 1, –1, and Z = A, B, Q}$$

the LKT can be written as:

$$\begin{cases} \beta_1^A = \rho_{1,0}^A\eta_0^A + \rho_{1,1}^A\eta_1^A + \rho_{1,-1}^A\eta_{-1}^A - \rho_{1,0}^Q\eta_0^Q - \rho_{1,1}^Q\eta_1^Q - \rho_{1,-1}^Q\eta_{-1}^Q \\ \beta_{-1}^A = \rho_{-1,0}^A\eta_0^A + \rho_{-1,1}^A\eta_1^A + \rho_{-1,-1}^A\eta_{-1}^A - \rho_{-1,0}^Q\eta_0^Q - \rho_{-1,1}^Q\eta_1^Q - \rho_{-1,-1}^Q\eta_{-1}^Q \\ \beta_1^B = -\rho_{1,0}^B\eta_0^B - \rho_{1,1}^B\eta_1^B - \rho_{1,-1}^B\eta_{-1}^B - \rho_{1,0}^Q\eta_0^Q - \rho_{1,1}^Q\eta_1^Q - \rho_{1,-1}^Q\eta_{-1}^Q \\ \beta_{-1}^B = -\rho_{-1,0}^B\eta_0^B - \rho_{-1,1}^B\eta_1^B - \rho_{-1,-1}^B\eta_{-1}^B - \rho_{-1,0}^Q\eta_0^Q - \rho_{-1,1}^Q\eta_1^Q - \rho_{-1,-1}^Q\eta_{-1}^Q \\ 0 = \rho_{0,0}^Q\eta_0^Q + \rho_{0,1}^Q\eta_1^Q + \rho_{0,-1}^Q\eta_{-1}^Q \end{cases}$$

Note that, since $\eta_{\pm1}^Q$ was expressed in terms of $\eta_{\pm1}^A$ and $\eta_{\pm1}^B$, and the last equation can be immediately solved for η_0^Q, the resulting system of four equations, containing $\eta_{\pm1}^A$, $\eta_{\pm1}^B$ and $\beta_{\pm1}^A$, $\beta_{\pm1}^B$, can be solved providing the current-voltage,i.e. η-β, characteristics of the input and output cables.

For what concerns the coefficients, a simple but cumbersome calculation leads to (with Z = A, B):

$$\rho_{1,0}^Z = \frac{1}{\sqrt{2}}\left(R_1^Z - R_2^Z\right) \qquad \rho_{1,1}^Z = R_1^Z + \frac{1}{2}R_2^Z \qquad \rho_{1,-1}^Z = -\frac{\sqrt{3}}{2}R_2^Z$$

$$\rho_{-1,0}^Z = \frac{1}{\sqrt{6}}\left(R_1^Z + R_2^Z - 2R_3^Z\right) \qquad \rho_{-1,1}^Z = \frac{1}{\sqrt{3}}\left(R_1^Z - \frac{1}{2}R_2^Z + R_3^Z\right) \qquad \rho_{-1,-1}^Z = \frac{1}{2}R_2^Z + R_3^Z$$

Furthermore, it can be easily demonstrated, since $[R^Q]$ is symmetric, than $\rho_{h,k}^Q = \rho_{k,h}^Q$. Therefore:

$$\rho_{0,0}^Q = \frac{1}{3}\left(R_{12} + R_{23} + R_{31}\right) \qquad \rho_{1,0}^Q = \frac{\sqrt{2}}{3}\left(R_{12} - \frac{R_{23} + R_{31}}{2}\right) \qquad \rho_{-1,0}^Q = \frac{1}{\sqrt{6}}\left(R_{23} - R_{31}\right)$$

$$\rho_{1,1}^Q = \frac{2}{3}\left(R_{12} + \frac{R_{23} + R_{31}}{4}\right) \qquad \rho_{-1,1}^Q = \frac{1}{2\sqrt{3}}\left(-R_{23} + R_{31}\right) \qquad \rho_{-1,-1}^Q = \frac{1}{2}\left(R_{23} + R_{31}\right)$$

To reduce the size of the solving system the last equation is solved for η_0^Q:

$$\eta_0^Q = -\frac{\rho_{0,1}^Q}{\rho_{0,0}^Q}\eta_1^Q - \frac{\rho_{0,-1}^Q}{\rho_{0,0}^Q}\eta_{-1}^Q$$

Consequently, defining (for h, k = ±1)

$$q_{k,h} = \rho_{k,h}^Q - \frac{\rho_{k,0}^Q\,\rho_{0,h}^Q}{\rho_{0,0}^Q}$$

and substituting, the following reduced system (containing only $\eta_{\pm1}^A$, $\eta_{\pm1}^B$ and v_{12}^A, v_{23}^A, v_{12}^B, v_{23}^B) is found:

$$\begin{cases} v_{12}^A\sqrt{\frac{3}{2}} = \rho_{1,0}^A\frac{I}{\sqrt{3}} + \rho_{1,1}^A\eta_1^A + \rho_{1,-1}^A\eta_{-1}^A - \frac{q_{1,1}}{2\sqrt{3}}\left[\sqrt{3}\left(\eta_1^B - \eta_1^A\right) - \left(\eta_{-1}^B - \eta_{-1}^A\right)\right] - \frac{q_{1,-1}}{2\sqrt{3}}\left[\left(\eta_1^B - \eta_1^A\right) + \sqrt{3}\left(\eta_{-1}^B - \eta_{-1}^A\right)\right] \\ \frac{v_{12}^A}{\sqrt{2}} + \sqrt{2}\,v_{23}^A = \rho_{-1,0}^A\frac{I}{\sqrt{3}} + \rho_{-1,1}^A\eta_1^A + \rho_{-1,-1}^A\eta_{-1}^A - \frac{q_{-1,1}}{2\sqrt{3}}\left[\sqrt{3}\left(\eta_1^B - \eta_1^A\right) - \left(\eta_{-1}^B - \eta_{-1}^A\right)\right] - \frac{q_{-1,-1}}{2\sqrt{3}}\left[\left(\eta_1^B - \eta_1^A\right) + \sqrt{3}\left(\eta_{-1}^B - \eta_{-1}^A\right)\right] \\ -v_{12}^B\sqrt{\frac{3}{2}} = \rho_{1,0}^B\frac{I}{\sqrt{3}} + \rho_{1,1}^B\eta_1^B + \rho_{1,-1}^B\eta_{-1}^B + \frac{q_{1,1}}{2\sqrt{3}}\left[\sqrt{3}\left(\eta_1^B - \eta_1^A\right) - \left(\eta_{-1}^B - \eta_{-1}^A\right)\right] + \frac{q_{1,-1}}{2\sqrt{3}}\left[\left(\eta_1^B - \eta_1^A\right) + \sqrt{3}\left(\eta_{-1}^B - \eta_{-1}^A\right)\right] \\ -\frac{v_{12}^B}{\sqrt{2}} - \sqrt{2}\,v_{23}^B = \rho_{-1,0}^B\frac{I}{\sqrt{3}} + \rho_{-1,1}^B\eta_1^B + \rho_{-1,-1}^B\eta_{-1}^B + \frac{q_{-1,1}}{2\sqrt{3}}\left[\sqrt{3}\left(\eta_1^B - \eta_1^A\right) - \left(\eta_{-1}^B - \eta_{-1}^A\right)\right] + \frac{q_{-1,-1}}{2\sqrt{3}}\left[\left(\eta_1^B - \eta_1^A\right) + \sqrt{3}\left(\eta_{-1}^B - \eta_{-1}^A\right)\right] \end{cases}$$

For what concerns the coefficients, a simple but cumbersome calculation leads to:

$$q_{1,1} = \frac{3}{2}\frac{R_{12}\left(R_{23} + R_{31}\right)}{R_{12} + R_{23} + R_{31}} \qquad q_{1,-1} = q_{-1,1} = -\frac{\sqrt{3}}{2}\frac{R_{12}\left(R_{23} - R_{31}\right)}{R_{12} + R_{23} + R_{31}} \qquad q_{-1,-1} = \frac{1}{2}\frac{R_{12}\left(R_{23} + R_{31}\right) + 4R_{23}R_{31}}{R_{12} + R_{23} + R_{31}}$$

A simpler form can be obtained introducing the star-equivalent resistances:

$$R_{1Y}=\frac{R_{12}R_{31}}{R_{12}+R_{23}+R_{31}}\qquad R_{2Y}=\frac{R_{12}R_{23}}{R_{12}+R_{23}+R_{31}}\qquad R_{3Y}=\frac{R_{23}R_{31}}{R_{12}+R_{23}+R_{31}}$$

And thus

$$q_{1,1}=\frac{3}{2}\left(R_{2Y}+R_{1Y}\right)\qquad q_{1,-1}=q_{-1,1}=-\frac{\sqrt{3}}{2}\left(R_{2Y}-R_{1Y}\right)\qquad q_{-1,-1}=\frac{R_{2Y}+R_{1Y}}{2}+2R_{3Y}$$

§ 3.2 –System solution in Laplace domain

Coupling the joint equations, written in the Laplace domain as follows

$$\begin{cases}
\tilde{v}_{12}^{A}\sqrt{\frac{3}{2}}=\rho_{1,0}^{A}\frac{\tilde{I}}{\sqrt{3}}+\rho_{1,1}^{A}\tilde{\eta}_{1}^{A}+\rho_{1,-1}^{A}\tilde{\eta}_{-1}^{A}-\frac{q_{1,1}}{2\sqrt{3}}\left[\sqrt{3}\left(\tilde{\eta}_{1}^{B}-\tilde{\eta}_{1}^{A}\right)-\left(\tilde{\eta}_{-1}^{B}-\tilde{\eta}_{-1}^{A}\right)\right]-\frac{q_{1,-1}}{2\sqrt{3}}\left[\left(\tilde{\eta}_{1}^{B}-\tilde{\eta}_{1}^{A}\right)+\sqrt{3}\left(\tilde{\eta}_{-1}^{B}-\tilde{\eta}_{-1}^{A}\right)\right]\\
\frac{\tilde{v}_{12}^{A}}{\sqrt{2}}+\sqrt{2}\,\tilde{v}_{23}^{A}=\rho_{-1,0}^{A}\frac{\tilde{I}}{\sqrt{3}}+\rho_{-1,1}^{A}\tilde{\eta}_{1}^{A}+\rho_{-1,-1}^{A}\tilde{\eta}_{-1}^{A}-\frac{q_{-1,1}}{2\sqrt{3}}\left[\sqrt{3}\left(\tilde{\eta}_{1}^{B}-\tilde{\eta}_{1}^{A}\right)-\left(\tilde{\eta}_{-1}^{B}-\tilde{\eta}_{-1}^{A}\right)\right]-\frac{q_{-1,-1}}{2\sqrt{3}}\left[\left(\tilde{\eta}_{1}^{B}-\tilde{\eta}_{1}^{A}\right)+\sqrt{3}\left(\tilde{\eta}_{-1}^{B}-\tilde{\eta}_{-1}^{A}\right)\right]\\
-\tilde{v}_{12}^{B}\sqrt{\frac{3}{2}}=\rho_{1,0}^{B}\frac{\tilde{I}}{\sqrt{3}}+\rho_{1,1}^{B}\tilde{\eta}_{1}^{B}+\rho_{1,-1}^{B}\tilde{\eta}_{-1}^{B}+\frac{q_{1,1}}{2\sqrt{3}}\left[\sqrt{3}\left(\tilde{\eta}_{1}^{B}-\tilde{\eta}_{1}^{A}\right)-\left(\tilde{\eta}_{-1}^{B}-\tilde{\eta}_{-1}^{A}\right)\right]+\frac{q_{1,-1}}{2\sqrt{3}}\left[\left(\tilde{\eta}_{1}^{B}-\tilde{\eta}_{1}^{A}\right)+\sqrt{3}\left(\tilde{\eta}_{-1}^{B}-\tilde{\eta}_{-1}^{A}\right)\right]\\
-\frac{\tilde{v}_{12}^{B}}{\sqrt{2}}-\sqrt{2}\,\tilde{v}_{23}^{B}=\rho_{-1,0}^{B}\frac{\tilde{I}}{\sqrt{3}}+\rho_{-1,1}^{B}\tilde{\eta}_{1}^{B}+\rho_{-1,-1}^{B}\tilde{\eta}_{-1}^{B}+\frac{q_{-1,1}}{2\sqrt{3}}\left[\sqrt{3}\left(\tilde{\eta}_{1}^{B}-\tilde{\eta}_{1}^{A}\right)-\left(\tilde{\eta}_{-1}^{B}-\tilde{\eta}_{-1}^{A}\right)\right]+\frac{q_{-1,-1}}{2\sqrt{3}}\left[\left(\tilde{\eta}_{1}^{B}-\tilde{\eta}_{1}^{A}\right)+\sqrt{3}\left(\tilde{\eta}_{-1}^{B}-\tilde{\eta}_{-1}^{A}\right)\right]
\end{cases}$$

with the input and output cables characteristics leads to (after reordering)

$$\begin{bmatrix}
\left(\frac{3s\underline{\tilde{\Delta}}}{2\gamma L}-\rho_{1,1}^{A}-\frac{q_{1,1}}{2}-\frac{q_{1,-1}}{2\sqrt{3}}\right) & \left(-\frac{\sqrt{3}s\underline{\tilde{\Delta}}}{2\gamma L}-\rho_{1,-1}^{A}+\frac{q_{1,1}}{2\sqrt{3}}-\frac{q_{1,-1}}{2}\right) & \left(\frac{q_{1,1}}{2}+\frac{q_{1,-1}}{2\sqrt{3}}\right) & \left(-\frac{q_{1,1}}{2\sqrt{3}}+\frac{q_{1,-1}}{2}\right)\\
\left(\frac{\sqrt{3}s\underline{\tilde{\Delta}}}{2\gamma L}-\rho_{-1,1}^{A}-\frac{q_{-1,1}}{2}-\frac{q_{-1,-1}}{2\sqrt{3}}\right) & \left(\frac{3s\underline{\tilde{\Delta}}}{2\gamma L}-\rho_{-1,-1}^{A}+\frac{q_{-1,1}}{2\sqrt{3}}-\frac{q_{-1,-1}}{2}\right) & \left(\frac{q_{-1,1}}{2}+\frac{q_{-1,-1}}{2\sqrt{3}}\right) & \left(-\frac{q_{-1,1}}{2\sqrt{3}}+\frac{q_{-1,-1}}{2}\right)\\
\left(\frac{q_{1,1}}{2}+\frac{q_{1,-1}}{2\sqrt{3}}\right) & \left(-\frac{q_{1,1}}{2\sqrt{3}}+\frac{q_{1,-1}}{2}\right) & \left(\frac{3s\underline{\tilde{\Delta}}}{2\gamma L}-\rho_{1,1}^{B}-\frac{q_{1,1}}{2}-\frac{q_{1,-1}}{2\sqrt{3}}\right) & \left(-\frac{\sqrt{3}s\underline{\tilde{\Delta}}}{2\gamma L}-\rho_{1,-1}^{B}+\frac{q_{1,1}}{2\sqrt{3}}-\frac{q_{1,-1}}{2}\right)\\
\left(\frac{q_{-1,1}}{2}+\frac{q_{-1,-1}}{2\sqrt{3}}\right) & \left(-\frac{q_{-1,1}}{2\sqrt{3}}+\frac{q_{-1,-1}}{2}\right) & \left(\frac{\sqrt{3}s\underline{\tilde{\Delta}}}{2\gamma L}-\rho_{-1,1}^{B}-\frac{q_{-1,1}}{2}-\frac{q_{-1,-1}}{2\sqrt{3}}\right) & \left(\frac{3s\underline{\tilde{\Delta}}}{2\gamma L}-\rho_{-1,-1}^{B}+\frac{q_{-1,1}}{2\sqrt{3}}-\frac{q_{-1,-1}}{2}\right)
\end{bmatrix}\begin{Bmatrix}\tilde{\eta}_{1}^{A}\\ \tilde{\eta}_{-1}^{A}\\ \tilde{\eta}_{1}^{B}\\ \tilde{\eta}_{-1}^{B}\end{Bmatrix}=\begin{Bmatrix}\rho_{1,0}^{A}\\ \rho_{-1,0}^{A}\\ \rho_{1,0}^{B}\\ \rho_{-1,0}^{B}\end{Bmatrix}\frac{\tilde{I}}{\sqrt{3}}$$

In order to symplify the matrix, let us perform the following operations: 1st) $(\text{Row1})\times\sqrt{3}+(\text{Row2})$; 2nd) $(\text{Row2})\times\sqrt{3}-(\text{Row1})$; 3rd) $(\text{Row3})\times\sqrt{3}+(\text{Row4})$; 4th) $(\text{Row4})\times\sqrt{3}-(\text{Row3})$. This leads to:

$$
\begin{bmatrix}
\left(\frac{2\sqrt{3}s\underline{\underline{\tilde{\Delta}}}}{\gamma L}-\rho^{A}_{1,1}\sqrt{3}-\rho^{A}_{-1,1}-\frac{q_{1,1}\sqrt{3}}{2}-q_{1,-1}-\frac{q_{-1,-1}}{2\sqrt{3}}\right) & \left(-\rho^{A}_{1,-1}\sqrt{3}-\rho^{A}_{-1,-1}+\frac{q_{1,1}}{2}-\frac{q_{1,-1}}{\sqrt{3}}-\frac{q_{-1,-1}}{2}\right) & \left(\frac{q_{1,1}\sqrt{3}}{2}+q_{1,-1}+\frac{q_{-1,-1}}{2\sqrt{3}}\right) & \left(-\frac{q_{1,1}}{2}+\frac{q_{1,-1}}{\sqrt{3}}+\frac{q_{-1,-1}}{2}\right) \\
\left(\rho^{A}_{1,1}-\rho^{A}_{-1,1}\sqrt{3}+\frac{q_{1,1}}{2}-\frac{q_{1,-1}}{\sqrt{3}}-\frac{q_{-1,-1}}{2}\right) & \left(\frac{2\sqrt{3}s\underline{\underline{\tilde{\Delta}}}}{\gamma L}+\rho^{A}_{1,-1}-\rho^{A}_{-1,-1}\sqrt{3}-\frac{q_{1,1}}{2\sqrt{3}}+q_{1,-1}-\frac{q_{-1,-1}\sqrt{3}}{2}\right) & \left(-\frac{q_{1,1}}{2}+\frac{q_{1,-1}}{\sqrt{3}}+\frac{q_{-1,-1}}{2}\right) & \left(\frac{q_{1,1}}{2\sqrt{3}}-q_{1,-1}+\frac{q_{-1,-1}\sqrt{3}}{2}\right) \\
\left(\frac{q_{1,1}\sqrt{3}}{2}+q_{1,-1}+\frac{q_{-1,-1}}{2\sqrt{3}}\right) & \left(-\frac{q_{1,1}}{2}+\frac{q_{1,-1}}{\sqrt{3}}+\frac{q_{-1,-1}}{2}\right) & \left(\frac{2\sqrt{3}s\underline{\underline{\tilde{\Delta}}}}{\gamma L}-\rho^{B}_{1,1}\sqrt{3}-\rho^{B}_{-1,1}-\frac{q_{1,1}\sqrt{3}}{2}-q_{1,-1}-\frac{q_{-1,-1}}{2\sqrt{3}}\right) & \left(-\rho^{B}_{1,-1}\sqrt{3}-\rho^{B}_{-1,-1}+\frac{q_{1,1}}{2}-\frac{q_{1,-1}}{\sqrt{3}}-\frac{q_{-1,-1}}{2}\right) \\
\left(-\frac{q_{1,1}}{2}+\frac{q_{1,-1}}{\sqrt{3}}+\frac{q_{-1,-1}}{2}\right) & \left(\frac{q_{1,1}}{2\sqrt{3}}-q_{1,-1}+\frac{q_{-1,-1}\sqrt{3}}{2}\right) & \left(\rho^{B}_{1,1}-\rho^{B}_{-1,1}\sqrt{3}+\frac{q_{1,1}}{2}-\frac{q_{1,-1}}{\sqrt{3}}-\frac{q_{-1,-1}}{2}\right) & \left(\frac{2\sqrt{3}s\underline{\underline{\tilde{\Delta}}}}{\gamma L}+\rho^{B}_{1,-1}-\rho^{B}_{-1,-1}\sqrt{3}-\frac{q_{1,1}}{2\sqrt{3}}+q_{1,-1}-\frac{q_{-1,-1}\sqrt{3}}{2}\right)
\end{bmatrix}
\begin{Bmatrix} \tilde{\eta}^{A}_{1} \\ \tilde{\eta}^{A}_{-1} \\ \tilde{\eta}^{B}_{1} \\ \tilde{\eta}^{B}_{-1} \end{Bmatrix}
=
\begin{Bmatrix} \rho^{A}_{1,0}\sqrt{3}+\rho^{A}_{-1,0} \\ \rho^{A}_{-1,0}\sqrt{3}-\rho^{A}_{1,0} \\ \rho^{B}_{1,0}\sqrt{3}+\rho^{B}_{-1,0} \\ \rho^{B}_{-1,0}\sqrt{3}-\rho^{B}_{1,0} \end{Bmatrix}
\frac{\tilde{I}}{\sqrt{3}}
$$

Fortunately, the involved coefficients can be simply expressed in terms of resistences, as follows

$$
\frac{q_{1,1}\sqrt{3}}{2}+q_{1,-1}+\frac{q_{-1,-1}}{2\sqrt{3}}=\frac{4R_{1Y}+R_{2Y}+R_{3Y}}{\sqrt{3}} \qquad -\frac{q_{1,1}}{2}+\frac{q_{1,-1}}{\sqrt{3}}+\frac{q_{-1,-1}}{2}=R_{3Y}-R_{2Y} \qquad \frac{q_{1,1}}{2\sqrt{3}}-q_{1,-1}+\frac{q_{-1,-1}\sqrt{3}}{2}=\sqrt{3}\left(R_{3Y}+R_{2Y}\right)
$$

(with Z = A, B)

$$
-\rho^{Z}_{1,1}\sqrt{3}-\rho^{Z}_{-1,1}=-\frac{4}{\sqrt{3}}R^{Z}_{1}-\frac{R^{Z}_{2}+R^{Z}_{3}}{\sqrt{3}} \qquad \rho^{Z}_{1,1}-\rho^{Z}_{-1,1}\sqrt{3}=-\rho^{Z}_{1,-1}\sqrt{3}-\rho^{Z}_{-1,-1}=R^{Z}_{2}-R^{Z}_{3}
$$

$$
\rho^{Z}_{1,-1}-\rho^{Z}_{-1,-1}\sqrt{3}=-\sqrt{3}\left(R^{Z}_{2}+R^{Z}_{3}\right) \qquad \rho^{Z}_{-1,0}+\rho^{Z}_{1,0}\sqrt{3}=-\sqrt{\frac{2}{3}}\left(2R^{Z}_{1}-R^{Z}_{2}-R^{Z}_{3}\right) \qquad -\rho^{Z}_{1,0}+\rho^{Z}_{-1,0}\sqrt{3}=\sqrt{2}\left(R^{Z}_{2}-R^{Z}_{3}\right)
$$

These expressions can be further simplified defining the following mean resistances

$$R_{mY} = \frac{R_{1Y} + R_{2Y} + R_{3Y}}{3} \qquad R_m^A = \frac{R_1^A + R_2^A + R_3^A}{3} \qquad R_m^B = \frac{R_1^B + R_2^B + R_3^B}{3}$$

Finally, multiplying all the rows by $-\gamma L/2\sqrt{3}$, the following system is obtained:

$$\begin{bmatrix} -s\underline{\tilde{\Delta}} + \frac{\gamma L}{2}\left(R_1^A + R_m^A + R_{1Y} + R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^A + R_{2Y} - R_3^A - R_{3Y}\right) & \frac{-\gamma L}{2}\left(R_{1Y} + R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y} - R_{2Y}\right) \\ \frac{-\gamma L}{2\sqrt{3}}\left(R_2^A + R_{2Y} - R_3^A - R_{3Y}\right) & -s\underline{\tilde{\Delta}} + \frac{\gamma L}{2}\left(R_2^A + R_3^A + R_{2Y} + R_{3Y}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y} - R_{2Y}\right) & \frac{-\gamma L}{2}\left(R_{2Y} + R_{3Y}\right) \\ \frac{-\gamma L}{2}\left(R_{1Y} + R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y} - R_{2Y}\right) & -s\underline{\tilde{\Delta}} + \frac{\gamma L}{2}\left(R_1^B + R_m^B + R_{1Y} + R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B + R_{2Y} - R_3^B - R_{3Y}\right) \\ \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y} - R_{2Y}\right) & \frac{-\gamma L}{2}\left(R_{2Y} + R_{3Y}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B + R_{2Y} - R_3^B - R_{3Y}\right) & -s\underline{\tilde{\Delta}} + \frac{\gamma L}{2}\left(R_2^B + R_3^B + R_{2Y} + R_{3Y}\right) \end{bmatrix} \begin{Bmatrix} \tilde{\eta}_1^A \\ \tilde{\eta}_{-1}^A \\ \tilde{\eta}_1^B \\ \tilde{\eta}_{-1}^B \end{Bmatrix} = \begin{Bmatrix} \frac{-\gamma L}{\sqrt{2}}\left(R_1^A - R_m^A\right) \\ \frac{-\gamma L}{\sqrt{6}}\left(R_2^A - R_3^A\right) \\ \frac{-\gamma L}{\sqrt{2}}\left(R_1^B - R_m^B\right) \\ \frac{-\gamma L}{\sqrt{6}}\left(R_2^B - R_3^B\right) \end{Bmatrix} \frac{\tilde{I}}{\sqrt{3}}$$

Now defining $\omega = s\underline{\tilde{\Delta}}(L, s)$, the solution of this system can be easily found following Cramer's rule:

$$\tilde{\eta}_p^Z(s) = \frac{\tilde{I}(s)}{\sqrt{3}} \frac{D_p^Z\left(s\underline{\tilde{\Delta}}(L, s)\right)}{D\left(s\underline{\tilde{\Delta}}(L, s)\right)}, \text{ with } Z = A, B \text{ and } p = \pm 1$$

where the D functions are defined as follows (all are determinants):

$$D_1^A(\omega)=\begin{vmatrix}
\frac{-\gamma L}{\sqrt{2}}\left(R_1^A-R_m^A\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^A+R_{2Y}-R_3^A-R_{3Y}\right) & \frac{-\gamma L}{2}\left(R_{1Y}+R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right)\\
\frac{-\gamma L}{\sqrt{6}}\left(R_2^A-R_3^A\right) & \frac{\gamma L}{2}\left(R_2^A+R_3^A+R_{2Y}+R_{3Y}\right)-\omega & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{-\gamma L}{2}\left(R_{2Y}+R_{3Y}\right)\\
\frac{-\gamma L}{\sqrt{2}}\left(R_1^B-R_m^B\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{\gamma L}{2}\left(R_1^B+R_m^B+R_{1Y}+R_{mY}\right)-\omega & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B+R_{2Y}-R_3^B-R_{3Y}\right)\\
\frac{-\gamma L}{\sqrt{6}}\left(R_2^B-R_3^B\right) & \frac{-\gamma L}{2}\left(R_{2Y}+R_{3Y}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B+R_{2Y}-R_3^B-R_{3Y}\right) & \frac{\gamma L}{2}\left(R_2^B+R_3^B+R_{2Y}+R_{3Y}\right)-\omega
\end{vmatrix}$$

$$D_{-1}^A(\omega)=\begin{vmatrix}
\frac{\gamma L}{2}\left(R_1^A+R_m^A+R_{1Y}+R_{mY}\right)-\omega & \frac{-\gamma L}{\sqrt{2}}\left(R_1^A-R_m^A\right) & \frac{-\gamma L}{2}\left(R_{1Y}+R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right)\\
\frac{-\gamma L}{2\sqrt{3}}\left(R_2^A+R_{2Y}-R_3^A-R_{3Y}\right) & \frac{-\gamma L}{\sqrt{6}}\left(R_2^A-R_3^A\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{-\gamma L}{2}\left(R_{2Y}+R_{3Y}\right)\\
\frac{-\gamma L}{2}\left(R_{1Y}+R_{mY}\right) & \frac{-\gamma L}{\sqrt{2}}\left(R_1^B-R_m^B\right) & \frac{\gamma L}{2}\left(R_1^B+R_m^B+R_{1Y}+R_{mY}\right)-\omega & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B+R_{2Y}-R_3^B-R_{3Y}\right)\\
\frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{-\gamma L}{\sqrt{6}}\left(R_2^B-R_3^B\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B+R_{2Y}-R_3^B-R_{3Y}\right) & \frac{\gamma L}{2}\left(R_2^B+R_3^B+R_{2Y}+R_{3Y}\right)-\omega
\end{vmatrix}$$

$$D_1^B(\omega)=\begin{vmatrix}
\frac{\gamma L}{2}\left(R_1^A+R_m^A+R_{1Y}+R_{mY}\right)-\omega & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^A+R_{2Y}-R_3^A-R_{3Y}\right) & \frac{-\gamma L}{\sqrt{2}}\left(R_1^A-R_m^A\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right)\\
\frac{-\gamma L}{2\sqrt{3}}\left(R_2^A+R_{2Y}-R_3^A-R_{3Y}\right) & \frac{\gamma L}{2}\left(R_2^A+R_3^A+R_{2Y}+R_{3Y}\right)-\omega & \frac{-\gamma L}{\sqrt{6}}\left(R_2^A-R_3^A\right) & \frac{-\gamma L}{2}\left(R_{2Y}+R_{3Y}\right)\\
\frac{-\gamma L}{2}\left(R_{1Y}+R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{-\gamma L}{\sqrt{2}}\left(R_1^B-R_m^B\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B+R_{2Y}-R_3^B-R_{3Y}\right)\\
\frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{-\gamma L}{2}\left(R_{2Y}+R_{3Y}\right) & \frac{-\gamma L}{\sqrt{6}}\left(R_2^B-R_3^B\right) & \frac{\gamma L}{2}\left(R_2^B+R_3^B+R_{2Y}+R_{3Y}\right)-\omega
\end{vmatrix}$$

$$D_{-1}^{B}(\omega)=\begin{vmatrix} \frac{\gamma L}{2}\left(R_1^A+R_m^A+R_{1Y}+R_{mY}\right)-\omega & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^A+R_{2Y}-R_3^A-R_{3Y}\right) & \frac{-\gamma L}{2}\left(R_{1Y}+R_{mY}\right) & \frac{-\gamma L}{\sqrt{2}}\left(R_1^A-R_m^A\right) \\ \frac{-\gamma L}{2\sqrt{3}}\left(R_2^A+R_{2Y}-R_3^A-R_{3Y}\right) & \frac{\gamma L}{2}\left(R_2^A+R_3^A+R_{2Y}+R_{3Y}\right)-\omega & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{-\gamma L}{\sqrt{6}}\left(R_2^A-R_3^A\right) \\ \frac{-\gamma L}{2}\left(R_{1Y}+R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{\gamma L}{2}\left(R_1^B+R_m^B+R_{1Y}+R_{mY}\right)-\omega & \frac{-\gamma L}{\sqrt{2}}\left(R_1^B-R_m^B\right) \\ \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{-\gamma L}{2}\left(R_{2Y}+R_{3Y}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B+R_{2Y}-R_3^B-R_{3Y}\right) & \frac{-\gamma L}{\sqrt{6}}\left(R_2^B-R_3^B\right) \end{vmatrix}$$

$$D(\omega)=\begin{vmatrix} \frac{\gamma L}{2}\left(R_1^A+R_m^A+R_{1Y}+R_{mY}\right)-\omega & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^A+R_{2Y}-R_3^A-R_{3Y}\right) & \frac{-\gamma L}{2}\left(R_{1Y}+R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) \\ \frac{-\gamma L}{2\sqrt{3}}\left(R_2^A+R_{2Y}-R_3^A-R_{3Y}\right) & \frac{\gamma L}{2}\left(R_2^A+R_3^A+R_{2Y}+R_{3Y}\right)-\omega & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{-\gamma L}{2}\left(R_{2Y}+R_{3Y}\right) \\ \frac{-\gamma L}{2}\left(R_{1Y}+R_{mY}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{\gamma L}{2}\left(R_1^B+R_m^B+R_{1Y}+R_{mY}\right)-\omega & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B+R_{2Y}-R_3^B-R_{3Y}\right) \\ \frac{-\gamma L}{2\sqrt{3}}\left(R_{3Y}-R_{2Y}\right) & \frac{-\gamma L}{2}\left(R_{2Y}+R_{3Y}\right) & \frac{-\gamma L}{2\sqrt{3}}\left(R_2^B+R_{2Y}-R_3^B-R_{3Y}\right) & \frac{\gamma L}{2}\left(R_2^B+R_3^B+R_{2Y}+R_{3Y}\right)-\omega \end{vmatrix}$$

It's apparent from the definition that $D(\omega)$ is a polynomial of fourth degree in ω, while $D_p^Z(\omega)$ are polynomials of third degree in ω. Moreover, considering the equation

$$D(\omega) = 0$$

which is clearly involved in inversion of the Laplace transformation of the solution found, it can be seen that it can be viewed as the secular equation for a symmetric real matrix D. Therefore, the eigenvalues are themselves real. To simplify the following calculation it is assumed that these eigenvalues (denoted by ω_1, ω_2, ω_3, ω_4) are distinct, i. e.

$$\frac{dD}{d\omega}(\omega)\neq 0 \quad \text{for } \omega = \omega_1, \omega_2, \omega_3, \omega_4$$

Moreover, Gershgorin theorem [4] provides bounds on them, i.e. for each eigenvalue one of the following relations is satisfied (with Z = A, B):

$$\left|\omega-\frac{\gamma L}{2}\left(R_1^Z+R_m^Z+R_{1Y}+R_{mY}\right)\right|\leq\frac{-\gamma L}{2\sqrt{3}}\left|R_2^Z+R_{2Y}-R_3^Z-R_{3Y}\right|+\frac{-\gamma L}{2}\left(R_{1Y}+R_{mY}\right)+\frac{-\gamma L}{2\sqrt{3}}\left|R_{3Y}-R_{2Y}\right|$$

$$\left|\omega-\frac{\gamma L}{2}\left(R_2^Z+R_3^Z+R_{2Y}+R_{3Y}\right)\right|\leq\frac{-\gamma L}{2\sqrt{3}}\left|R_2^Z+R_{2Y}-R_3^Z-R_{3Y}\right|+\frac{-\gamma L}{2\sqrt{3}}\left|R_{3Y}-R_{2Y}\right|+\frac{-\gamma L}{2}\left(R_{2Y}+R_{3Y}\right)$$

Before going on with the back-transformation of the solution found, let us study the equation

$$s\tilde{\underline{\Delta}}(L,s)=\omega \qquad \Rightarrow \qquad L\sqrt{s\gamma\lambda}\,\frac{\cos\left(L\sqrt{s\gamma\lambda}\right)}{\sin\left(L\sqrt{s\gamma\lambda}\right)}=\omega$$

The change of variable $x = L\sqrt{s\gamma\lambda}$ leads to the equation

$$\cot x = \frac{\omega}{x}$$

where x can be considered as a real positive unknown, since $s = \frac{1}{\lambda\gamma}\left(\frac{x}{L}\right)^2$

As it's shown in the figure (where it's assumed that $\omega < 0$), the solutions of this equation, denoted by $\xi_k(\omega)$ for k = 0, 1, 2, …, are bounded as

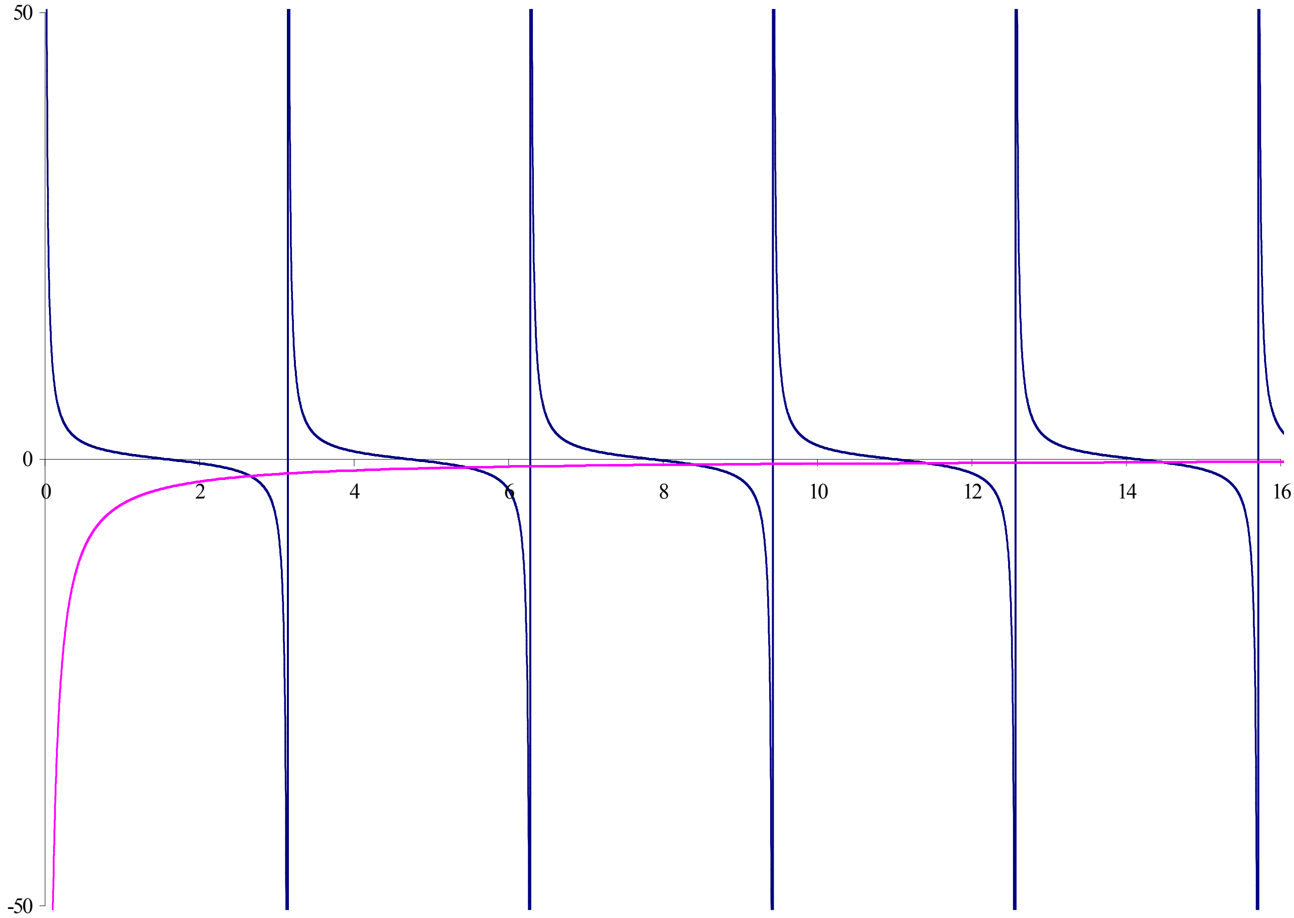
$$\left(k+\frac{1}{2}\right)\pi < \xi_k(\omega) < (k+1)\pi\text{, for } \omega < 0 \text{ and } k = 0, 1, 2, \ldots$$

If ω is positive a similar situation occurs; the main difference is the disappearance of the root $\xi_0(\omega)$ if $\omega >$ 1. The other solutions, denoted as before by $\xi_k(\omega)$ for k = 1, 2, …, are bounded as

$$k\pi < \xi_k(\omega) < \left(k+\frac{1}{2}\right)\pi\text{, for } \omega > 0 \text{ and } k = 1, 2, \ldots$$

In any way, it results that: $\lim_{k\to\infty}\left[\xi_k(\omega)-\left(k+\frac{1}{2}\right)\pi\right]=0$.

Note that the special case $\omega = 0$, that leads to an explicit solution for the $\xi_k(\omega)$, is meaningless, since a null eigenvalue of the matrix **D** implies its singularity and thus that the problem is ill-posed.



§ 3.3 –Solution in time domain

Defining $\widetilde{\Psi}_p^Z(s) = \dfrac{D_p^Z\left(s\underline{\widetilde{\Delta}}(L,s)\right)}{D\left(s\underline{\widetilde{\Delta}}(L,s)\right)}$, with Z = A, B and p = ±1

the solution in the Laplace domain is written as:

$$\widetilde{\eta}_p^Z(s) = \frac{\widetilde{I}(s)}{\sqrt{3}}\widetilde{\Psi}_p^Z(s), \text{ with Z = A, B and p = ±1}$$

and formally inverted in the time domain as follows

$$\eta_p^Z(t) = \int_0^t d\tau \, \frac{I(t-\tau)}{\sqrt{3}} \Psi_p^Z(\tau), \text{ with Z = A, B and p = ±1}$$

To perform the inversion of the functions $\widetilde{\Psi}_p^Z(s)$ note that they satisfy the Jordan Lemma [5]. Thus, with the assumptions made, this leads to:

$$\Psi_p^Z(t) = \sum_{h=1}^{4} \frac{D_p^Z(\omega_h)}{D'(\omega_h)} \sum_{k=u(\omega_h)}^{\infty} \frac{e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2}}{\frac{d}{ds}\left[s\underline{\widetilde{\Delta}}(L,s)\right]_{s=\frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2}}, \text{ with Z = A, B and p = ±1}$$

where the (infinite) residues have been directly evaluated with the aid of the De L'Hopital rule, since the eingenvalues was assumed to be distinct. The function u(ω) is defined as 0, if ω < 1 and 1 otherwise. The involved differential is simply evaluated as:

$$\frac{d}{ds}\left[s\underline{\widetilde{\Delta}}(L,s)\right]_{s=\frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2} = \frac{d}{ds}\left[L\sqrt{s\gamma\lambda}\right]_{s=\frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2} \cdot \frac{d}{dx}\left[x\cot x\right]_{x=\xi_k(\omega_h)} =$$

$$= \left[\frac{L}{2}\sqrt{\frac{\gamma\lambda}{s}}\right]_{s=\frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2} \cdot \left[\cot x - x\left(1+\cot^2 x\right)\right]_{x=\xi_k(\omega_h)} =$$

$$= \frac{\gamma\lambda L^2}{2}\left[\frac{\cos\xi_k(\omega_h)}{\xi_k(\omega_h)\,\text{sen}\,\xi_k(\omega_h)} - \frac{1}{\text{sen}^2\,\xi_k(\omega_h)}\right] =$$

$$= \frac{\gamma\lambda L^2}{2}\left[\frac{\cos\xi_k(\omega_h)\,\text{sen}\,\xi_k(\omega_h) - \xi_k(\omega_h)}{\xi_k(\omega_h)\,\text{sen}^2\,\xi_k(\omega_h)}\right]$$

(taking into account that $\dfrac{\text{sen}\,\xi_k(\omega_h)}{\xi_k(\omega_h)} = \dfrac{\cos\xi_k(\omega_h)}{\omega_h}$) and finally

$$\Psi_p^Z(t) = \frac{2}{\gamma\lambda L^2}\sum_{h=1}^{4} \frac{D_p^Z(\omega_h)}{D'(\omega_h)} \sum_{k=u(\omega_h)}^{\infty} \frac{\xi_k(\omega_h)\,\text{sen}^2\,\xi_k(\omega_h)\, e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2}}{\cos\xi_k(\omega_h)\,\text{sen}\,\xi_k(\omega_h) - \xi_k(\omega_h)}, \text{ with Z = A, B and p = ±1}$$

Note that, since its poles characterize an analytical function, of the functions $\widetilde{\Psi}_p^Z(s)$ admits also the following representation:

$$\widetilde{\Psi}_p^Z(s) = \frac{2}{\gamma\lambda L^2}\sum_{h=1}^{4} \frac{D_p^Z(\omega_h)}{D'(\omega_h)} \sum_{k=u(\omega_h)}^{\infty} \frac{\xi_k(\omega_h)\,\text{sen}^2\,\xi_k(\omega_h)}{\cos\xi_k(\omega_h)\,\text{sen}\,\xi_k(\omega_h) - \xi_k(\omega_h)} \frac{1}{s - \frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2},$$

with Z = A, B and p = ±1

To complete the treatment, let us represent the currents in the inlet and outlet cables (through their η-components). Consider first the inlet cable. It was shown before that

$$\psi_p(x,t) = \frac{1}{\gamma L}\int_0^t d\tau\, \eta_p'^A(t-\tau)\underline{\Delta}(x,\tau) = \frac{1}{\gamma L}\left[\eta_p'^A(\cdot) * \underline{\Delta}(x,\cdot)\right](t) \qquad \text{with } t>0,\ 0<x<L \text{ and } p=\pm 1$$

$$\eta_p(x,t) = \gamma\int_0^x dx\, \psi_p(x,t) \qquad \text{with } t>0,\ 0<x<L \text{ and } p=\pm 1$$

where the star denotes the convolution operation. Furthermore, differentiation of a previous result leads to [6]:

$$\eta_p'^A(t) = \int_0^t d\tau\, \frac{I'(t-\tau)}{\sqrt{3}}\Psi_p^A(\tau) = \left[\frac{I'(\cdot)}{\sqrt{3}} * \Psi_p^A(\cdot)\right](t) \qquad \text{with } p=\pm 1$$

and substituting:

$$\psi_p(x,t) = \frac{1}{\gamma L}\left[\frac{I'(\cdot)}{\sqrt{3}} * \Psi_p^A(\cdot) * \underline{\Delta}(x,\cdot)\right](t) \qquad \text{with } t>0,\ 0<x<L \text{ and } p=\pm 1$$

Substituting again:

$$\eta_p(x,t) = \left[\frac{I'(\cdot)}{\sqrt{3}} * \left\{\frac{1}{L}\int_0^x dx\, \Psi_p^A(\cdot) * \underline{\Delta}(x,\cdot)\right\}\right](t) \qquad \text{with } t>0,\ 0<x<L \text{ and } p=\pm 1$$

Therefore, we begin evaluating the convolution of $\Psi_p^A(t)$ with $\underline{\Delta}(x,t)$:[(o)]

$$\left[\Psi_p^A(\cdot) * \underline{\Delta}(x,\cdot)\right](t) =$$

$$= \frac{2}{\gamma\lambda L^2}\sum_{h=1}^{4}\frac{D_p^A(\omega_h)}{D'(\omega_h)}\sum_{k=u(\omega_h)}^{\infty}\frac{\xi_k(\omega_h)\,\mathrm{sen}^2\,\xi_k(\omega_h)}{\cos\xi_k(\omega_h)\,\mathrm{sen}\,\xi_k(\omega_h)-\xi_k(\omega_h)}\sum_{n=-\infty}^{+\infty}\cos\left(\frac{n\pi x}{L}\right)\cos(n\pi)\left[e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2} * e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}\right] =$$

$$= \frac{2}{\gamma\lambda L^2}\sum_{h=1}^{4}\frac{D_p^A(\omega_h)}{D'(\omega_h)}\sum_{k=u(\omega_h)}^{\infty}\frac{\xi_k(\omega_h)\,\mathrm{sen}^2\,\xi_k(\omega_h)}{\cos\xi_k(\omega_h)\,\mathrm{sen}\,\xi_k(\omega_h)-\xi_k(\omega_h)}\sum_{n=-\infty}^{+\infty}\cos\left(\frac{n\pi x}{L}\right)\cos(n\pi)\frac{e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2} - e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}}{\frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2-\frac{1}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2} =$$

$$= \frac{2}{\gamma\lambda L^2}\sum_{h=1}^{4}\frac{D_p^A(\omega_h)}{D'(\omega_h)}\sum_{k=u(\omega_h)}^{\infty}\frac{\xi_k(\omega_h)\,\mathrm{sen}^2\,\xi_k(\omega_h)\,e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2}}{\cos\xi_k(\omega_h)\,\mathrm{sen}\,\xi_k(\omega_h)-\xi_k(\omega_h)}\left\{\sum_{n=-\infty}^{+\infty}\cos\left(\frac{n\pi x}{L}\right)\cos(n\pi)\frac{1}{\frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2-\frac{1}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}\right\} +$$

$$+ \sum_{n=-\infty}^{+\infty}\cos\left(\frac{n\pi x}{L}\right)\cos(n\pi)\left\{\frac{2}{\gamma\lambda L^2}\sum_{h=1}^{4}\frac{D_p^A(\omega_h)}{D'(\omega_h)}\sum_{k=u(\omega_h)}^{\infty}\frac{\xi_k(\omega_h)\,\mathrm{sen}^2\,\xi_k(\omega_h)}{\cos\xi_k(\omega_h)\,\mathrm{sen}\,\xi_k(\omega_h)-\xi_k(\omega_h)}\frac{1}{\frac{1}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2-\frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2}\right\}e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}$$

Note that the series in the brackets was previously developed as representations of $\tilde{\Psi}_p^A(s)$ and $\underline{\tilde{\Delta}}(x,s)$. Therefore

(o) Note that $\left[e^{\alpha t} * e^{\beta t}\right] = \int_0^t d\tau\, e^{\alpha\tau}e^{\beta(t-\tau)} = \frac{e^{\alpha t}-e^{\beta t}}{\alpha-\beta}$, provided $\alpha \neq \beta$

$$\left[\Psi_p^A(\cdot) * \underline{\Delta}(x,\cdot)\right](t) = \frac{2}{\gamma\lambda L^2} \sum_{h=1}^{4} \frac{D_p^A(\omega_h)}{D'(\omega_h)} \sum_{k=u(\omega_h)}^{\infty} \frac{\xi_k(\omega_h)\,\mathrm{sen}^2\,\xi_k(\omega_h)\, e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2}}{\cos\xi_k(\omega_h)\,\mathrm{sen}\,\xi_k(\omega_h) - \xi_k(\omega_h)}\, \tilde{\underline{\Delta}}\left(x, \frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2\right) +$$

$$+ \sum_{n=-\infty}^{+\infty} \cos\left(\frac{n\pi x}{L}\right) \cos(n\pi)\, \tilde{\Psi}_p^A\left(\frac{1}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2\right) e^{\frac{t}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2}$$

Now taking into account that $\tilde{\underline{\Delta}}\left(x, \frac{1}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2\right) = \frac{\gamma\lambda L^2}{\xi_k(\omega_h)} \frac{\cos[\xi_k(\omega_h)\cdot x/L]}{\mathrm{sen}\,\xi_k(\omega_h)}$ and

$$s\,\tilde{\underline{\Delta}}(L,s) = \begin{cases} \infty, & \text{for } s = \frac{1}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2, n \neq 0 \\ 1, & \text{for } s = 0 \end{cases} \quad \Rightarrow \quad \tilde{\Psi}_p^A(s) = \begin{cases} 0, & \text{for } s = \frac{1}{\lambda\gamma}\left(\frac{n\pi}{L}\right)^2, n \neq 0 \\ \frac{D_p^A(1)}{D(1)}, & \text{for } s = 0 \end{cases}$$

gives

$$\left[\Psi_p^A(\cdot) * \underline{\Delta}(x,\cdot)\right](t) = 2\sum_{h=1}^{4} \frac{D_p^A(\omega_h)}{D'(\omega_h)} \sum_{k=u(\omega_h)}^{\infty} \frac{\mathrm{sen}\xi_k(\omega_h)\cos[\xi_k(\omega_h)\cdot x/L]}{\cos\xi_k(\omega_h)\mathrm{sen}\xi_k(\omega_h) - \xi_k(\omega_h)} e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2} + \frac{D_p^A(1)}{D(1)}$$

Spatial integration now gives

$$\frac{1}{L}\int_0^x dx\, \left[\Psi_p^A(\cdot) * \underline{\Delta}(x,\cdot)\right](t) = 2\sum_{h=1}^{4} \frac{D_p^A(\omega_h)}{D'(\omega_h)} \sum_{k=u(\omega_h)}^{\infty} \frac{\frac{\mathrm{sen}\xi_k(\omega_h)}{\xi_k(\omega_h)}\mathrm{sen}[\xi_k(\omega_h)\cdot x/L]}{\cos\xi_k(\omega_h)\mathrm{sen}\xi_k(\omega_h) - \xi_k(\omega_h)} e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2} + \frac{x}{L}\frac{D_p^A(1)}{D(1)} =$$

$$= 2\sum_{h=1}^{4} \frac{D_p^A(\omega_h)}{\omega_h D'(\omega_h)} \sum_{k=u(\omega_h)}^{\infty} \frac{\cos\xi_k(\omega_h)\mathrm{sen}[\xi_k(\omega_h)\cdot x/L]}{\cos\xi_k(\omega_h)\mathrm{sen}\xi_k(\omega_h) - \xi_k(\omega_h)} e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2} + \frac{x}{L}\frac{D_p^A(1)}{D(1)}$$

Finally, defining

$$F_p^A(x,t) = 2\sum_{h=1}^{4} \frac{D_p^A(\omega_h)}{\omega_h D'(\omega_h)} \sum_{k=u(\omega_h)}^{\infty} \frac{\cos\xi_k(\omega_h)\,\mathrm{sen}[\xi_k(\omega_h)\cdot x/L]}{\cos\xi_k(\omega_h)\,\mathrm{sen}\,\xi_k(\omega_h) - \xi_k(\omega_h)} e^{\frac{t}{\lambda\gamma}\left(\frac{\xi_k(\omega_h)}{L}\right)^2}$$

leads to

$$\eta_p(x,t) = \frac{x}{L}\frac{D_p^A(1)}{D(1)}\frac{I(t)}{\sqrt{3}} + \int_0^t d\tau\, \frac{I'(t-\tau)}{\sqrt{3}} F_p^A(x,\tau) \qquad \text{with } t > 0,\ 0 < x < L \text{ and } p = \pm 1$$

For what concerns the outlet cable, since it's formally symmetric to the inlet one, the solution for $\eta_{\pm 1}$, can be obtained changing A with B and considering 2L – x instead of x, as follows

$$\eta_p(x,t) = \frac{2L - x}{L}\frac{D_p^B(1)}{D(1)}\frac{I(t)}{\sqrt{3}} + \int_0^t d\tau\, \frac{I'(t-\tau)}{\sqrt{3}} F_p^B(2L - x,\tau) \qquad \text{with } t > 0,\ L < x < 2L \text{ and } p = \pm 1$$

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