Determination of X-Points in Magnetic Induction Fields

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***Abstract* - A sufficient condition for the existence of a node (i.e., an X-point) in lines of a steady-state magnetic induction field is proved. The theoretical approach is developed for two-dimensional fields. The problem of obtaining an X-point in a magnetic induction field is encountered in toroidal devices (tokamaks) for controlled thermonuclear fusion experiments. The proposed method is capable to identify possible X-point locations through the knowledge of field variables which can be calculated by both analytical and numerical methods.**

INTRODUCTION

In some applications, such as for instance in the design of a toroidal device for experiments on controlled thermonuclear fusion with magnetic confinement (tokamak), the problem of obtaining an X-point in lines of a magnetic induction field $\boldsymbol{B}$ can be encountered. In particular, in the divertor of a tokamak such an X-point should be obtained [1]. We can assume that field variables may satisfy some conditions at the X-point, the fulfilment of which assures the existence of an X-point in the field $\boldsymbol{B}$ of the device under consideration. The knowledge of such conditions and the possibility to verify their satisfaction could be very useful when attempting either to ascertain or maintain the presence of an X-point. This holds true both for design and operation.

In this paper, an analytical approach to the problem is provided. The proposed theoretical analysis affords sufficient conditions relating field variables at points which are X-point locations. Therefore, the provided conditions allows one to identify and maintain possible X-point locations. The analysis is carried out with reference to toroidal devices. These devices are axisymmetric systems. As a consequence, we can deal with two-dimensional magnetic induction fields (e.g., systems with a translational symmetry and axisymmetric systems).

THEORETICAL ANALYSIS

In two-dimensional problems involving a translational symmetry of the magnetic field, the magnetic vector potential $\boldsymbol{A}$ has a component along the translational axis only. Assuming a cartesian orthogonal coordinate system with the z-axis coinciding with the translational axis, we can write the vector potential component $A_z = \Psi$, which is a function of x and y alone. The scalar function Ψ is related to the magnetic induction field $\boldsymbol{B}$ by the following equation

$$\boldsymbol{B}(x,y) = \nabla \times \left[\Psi(x,y)\,\boldsymbol{k}\right] = \nabla\Psi(x,y) \times \boldsymbol{k}\,, \tag{1}$$

where $\boldsymbol{k}$ is the unit vector in the positive z-direction. From (1), it follows that the equation of the magnetic induction lines (flux lines) in the xy-plane is given by $\Psi(x, y)$ = const. The curl of the magnetic induction field is then given by

$$\nabla \times \boldsymbol{B} = -\nabla^2\Psi(x,y)\,\boldsymbol{k}\,. \tag{2}$$

For steady-state magnetic fields, we can also write the Ampère's law in the form

$$\nabla \times \boldsymbol{B} = \mu_0 J(x,y)\,\boldsymbol{k}\,, \tag{3}$$

where J is the z-directed component of the current density vector, and μ_0 is the permittivity of vacuum.

In the final paper, we will prove the following theorem.

Theorem

For a point (x_0, y_0) of a flux line $\Psi(x,y) = \text{const.}$ to be a node (i.e., an X-point), it is sufficient that the following conditions are satisfied in (x_0, y_0):

i) $B(x_0, y_0) = 0$ [or $B^2(x_0, y_0) = 0$] .

ii) $B^2(x,y)$ has a local minimum at the point (x_0, y_0).

iii) $J(x_0, y_0) = 0$.

We can consider also axisymmetric magnetic fields. We can assume a cylindrical coordinate system r, ϕ, z, with the z-axis coincident with the axis of symmetry. The magnetic vector potential $\boldsymbol{A}$ has only a ϕ-directed component $A_\phi(r, z)$. In this case, the scalar function $\Psi(r, z)$ is defined as [1]

$$\Psi(r,z) = 2\pi r A_\phi(r,z). \tag{4}$$

Equations. (1) and (2)-(3) become

$$\boldsymbol{B} = \nabla\Psi \times \frac{\boldsymbol{u}_\phi}{2\pi r}. \tag{5}$$

and

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{\partial^2\Psi}{\partial z^2} - \frac{1}{r}\frac{\partial\Psi}{\partial r} = -2\pi\mu_0 r J(r,z), \tag{6}$$

where $\boldsymbol{u}_\phi$ is the unit vector in the direction of increasing values of ϕ, and J is the ϕ-directed component of the current density vector. The proof of the theorem introduced holds true also in this case. In fact, the equations obtained in the proof of the theorem are still valid, when variables x and y are replaced by the variables r and z, respectively.

Furthermore, in the final paper we will show that at an X-point the two branches of the flux line intersect each other at a right angle.

NUMERICAL EXAMPLES

We firstly consider a simple example to illustrate the validity of the proposed sufficient condition. This example has an analytical solution. We consider the magnetic induction field $\boldsymbol{B}$ produced by two infinitely long parallel wires carrying steady-state currents of the same magnitude and direction, I. A cross section of the system, which has a translational symmetry, is shown in Fig. 1. The cross-sectional plane is the xy-plane of the chosen orthogonal cartesian system shown in the figure.

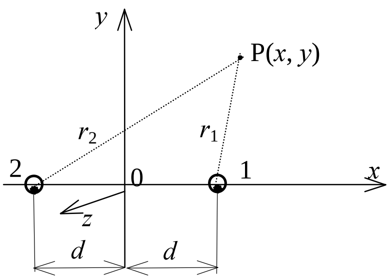

Fig. 1. Cross-sectional view of two infinitely long parallel wires.

From (1) the corresponding function B^2 can be written as

$$B^2(x,y) = \left(\frac{\partial\Psi}{\partial x}\right)^2 + \left(\frac{\partial\Psi}{\partial y}\right)^2, \tag{7}$$

where the flux function $\Psi(x, y)$ is given by [2]

$$\begin{aligned}\Psi(x, y) &= k(\log r_1 + \log r_2) = \\ &= k\left[\log\sqrt{(x-d)^2+y^2} + \log\sqrt{(x+d)^2+y^2}\right].\end{aligned} \tag{8}$$

In (8) d is half the distance between wires, and $k = -\mu_0 I / 2\pi$. The first-order derivatives of (8) at the point (0, 0) are equal to zero

$$\left(\frac{\partial\Psi}{\partial x}\right)_{0,0} = 0, \qquad \left(\frac{\partial\Psi}{\partial y}\right)_{0,0} = 0\,. \tag{9}$$

From (7) and (9), it follows that condition i) is satisfied at the point (0, 0).

Calculating the second-order derivatives of (8) at the point (0, 0) yields

$$\left(\frac{\partial^2\Psi}{\partial x^2}\right)_{0,0} = -\frac{2k}{d^2}, \tag{10}$$

$$\left(\frac{\partial^2\Psi}{\partial y^2}\right)_{0,0} = \frac{2k}{d^2}, \tag{11}$$

$$\left(\frac{\partial^2\Psi}{\partial x \partial y}\right)_{0,0} = 0\,. \tag{12}$$

From (9)-(12), it can be easily verified that at the point (0, 0) the first-order derivatives of the function B^2, given by (7), are null and that its Hessian is positive definite. This is a sufficient condition so that the function B^2 has a local minimum at the point (0, 0) [3], i.e., condition ii) is satisfied at this point. Furthermore, it is $J(0, 0) = 0$.

As a consequence, conditions of the theorem are simultaneously satisfied at the point (0, 0) which, therefore, is an X-point. The resulting flux plot is depicted in Fig. 2.

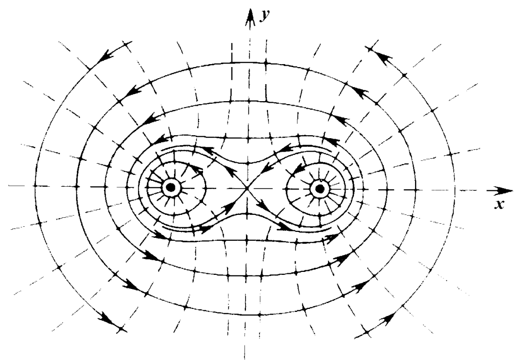

Fig. 2. Flux plot for configuration shown in Fig. 1.

In the final paper the proposed theorem will be applied to identify X-points in magnetic fields calculated through numerical methods.

CONCLUSIONS

A theorem which gives sufficient conditions for determining X-points in two-dimensional magnetic induction fields has been proved. The validity of the theorem has been demonstrated on a simple example. The theorem provides easy determination of X-points from the knowledge of the values of field variables in the neighbourhood of the isolate null points of the field.

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