Energy-Based Variational Model for Vector Magnetic Hysteresis

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1. Magnetic levitation systems
2. Coated conductors with soft magnetic substrates.
3. Metamaterials used for:
   - magnetic cloaking;
   - distant invisible transfer of magnetic field;
   - ...
Models for hysteresis in type-II superconductors are better developed.

Modeling magnetic materials in hybrid systems: a constant (finite or infinite) permeability $\mu_r$ or a nonlinear function $\mu_r(h)$.

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Modeling magnetic materials in hybrid systems: a constant (finite or infinite) permeability $\mu_r$ or a nonlinear function $\mu_r(h)$.

Should the magnetic hysteresis be neglected? Not always.

**Example:** coated conductors with a Ni substrate.

- The characteristic magnetic field for a sc strip (Brandt and Indenbom, 1993), $H_c = J_c/\pi$, is of the order $10^3 - 10^4$ A/m.
- Coercivity of a Ni substrate is similar, about 6000 A/m (Ijaduola et al. 2004).
Wanted: a model for magnetic hysteresis

Requirements: the model should

- be vectorial:

![Diagram showing b lines for a weak (left) and strong (right) external field.]

Fig: \( b \) lines for a weak (left) and strong (right) external field.

- predict the magnetization losses;
- be able to account for material anisotropy;
- be included into the Maxwell equations as a local constitutive relation with memory.
Models for quasistationary hysteresis

The most popular macroscopic models are

- **The Preisach model (1935):** a black-box-type method for storing, and using for interpolation, a vast amount of experimental data.

- The Jiles-Atherton model (1984): much simpler to implement but can show a nonphysical behaviour and needs a patch, not sufficiently accurate.

The models are scalar; their vectorial versions exist but are not physically justified too.

Energy loss: can be estimated only for closed loops.
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Other disadvantages:

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- Energy loss: can be estimated only for closed loops.
The Bergqvist (1997) model:

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- intrinsically vectorial;
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Subsequent works: Bergqvist et al. (97, 04, 14) Henrotte, Hameyer, Steentjes, Geuzaine, … (06, 12, 13, 14).

Weak points: derivation and numerical implementation
An approximation to make the magnetization update explicit. Should be avoided in the vectorial case.
This talk:

- A simplified energy-based model:
  - derivation;
  - variational structure;
  - numerical algorithm.
- The composite Bergqvist model.
- Identification of the parameters in the model.
- Coupling with the Maxwell equations: an example.
A simplified model: Assumption 1

A1: The density of magnetic field energy in a magnetic material

\[ W = \frac{1}{2} \mu_0 h^2 + U(m) \] changes as

\[ \dot{W} = h \cdot \dot{b} - |r \dot{m}|, \]

where \( b = \mu_0 (h + m) \),

\( h \cdot \dot{b} \) - the rate of magnetic field work, and

\( |r \dot{m}| \) - the rate of dissipation due to the irreversible movement of magnetic domain walls.

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This yields

\[ \mu_0 h \cdot \dot{h} + \nabla U(m) \cdot \dot{m} = \mu_0 h \cdot (\dot{h} + \dot{m}) - |r \dot{m}| \]

or

\[ (h - f(m)) \cdot \dot{m} = |k \dot{m}|, \]

where \( f(m) = \frac{1}{\mu_0} \nabla U(m) \) and \( k = \frac{1}{\mu_0} r \).
Bergqvist presented the magnetic field as a sum,
\[ h = h_r + h_i, \]
where
the field \( h_r = f(m) = \frac{1}{\mu_0} \nabla U(m) \) is called reversible, because the
magnetic work it delivers is fully converted into internal energy.
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The energy-based relation \( (h - f(m)) \cdot \dot{m} = |k\dot{m}| \) becomes
\[ h_i \cdot \dot{m} = |k\dot{m}|. \]

**A2:** For an isotropic material this relation holds if the following “dry-friction law” is postulated:
\[ |h_i| \leq k; \]
if \( |h_i| < k \) then \( \dot{m} = 0 \); if \( \dot{m} \neq 0 \) it has the direction of \( h_i \).
Equivalent formulation of this law is:

\[ h_i \in \tilde{K} := \{ u \in \mathbb{R}^3 : |k^{-1}u| \leq 1 \} \quad \text{and} \quad \dot{m} \cdot (u - h_i) \leq 0 \quad \text{for any} \quad u \in K. \]
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Written in this form, the constitutive relation:

- agrees with the "dry friction law" also in the anisotropic case;
- means that \( \dot{m} \) is a subgradient of the indicator function of the set \( \tilde{K} \) at the point \( h_i \) (the variational structure);
The variational structure

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- means that \( \dot{m} \) is a subgradient of the indicator function of the set \( \tilde{K} \) at the point \( h_i \) (the variational structure);
- is similar to the relations between:
  - the rate of plastic deformations and stress in elasto-plasticity with the yield stress \( k \);
  - the electric field and current density in type-II superconductivity models.
S1. Let $h(t)$ be given.
If $h_i \in \tilde{K}$ then $h_r = h(t) - h_i$ belongs to the set

$$K(t) := \{u \in \mathbb{R}^3 : |k^{-1}(h(t) - u)| \leq 1\}.$$

The model can be reformulated for $h_r$,

Find $h_r \in K(t)$ and $m$ such that

$$\dot{m} \cdot (u - h_r) \geq 0 \text{ for any } u \in K(t),$$

and discretized in time,

Find $h_r \in K(t)$ and $m$ are such that

$$(m - \hat{m}) \cdot (u - h_r) \geq 0 \text{ for any } u \in K(t).$$

Finally, we express $m$ via $h_r$. 
S2. Since \( h_r \parallel m \) we set \( m = M_{an}(h_r) \frac{h_r}{h_r} \), where \( M_{an} \) is a nondecreasing anhysteretic function s.t. \( M_{an}(0) = 0 \), and define

\[
S(u) = \int_0^u M_{an}(s)ds.
\]

Then \( m = \nabla S(h_r) \).
Numerical solution: Step 2

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$$S(u) = \int_0^u M_{an}(s)ds.$$ 

Then $m = \nabla S(h_r)$. On each time level the problem becomes

Find $h_r \in K(t)$ such that

$$(\nabla S(h_r) - \hat{m}) \cdot (u - h_r) \geq 0$$

for any $u \in K(t)$. 

Equivalently, $h_r$ is the unique solution to the optimization problem

$$\min_{u \in K(t)} \{S(u) - \hat{m} \cdot u\}$$

which we could solve very efficiently (2-3 iterations). Bergqvist and other authors after him used instead a projection: $h_r = Proj_{K(t)}\{\hat{h}_r\}$ (explicit formulae).
Scalar example: \( h = (H_m \sin t, 0) \).

\( M_{an} \) and \( k \) are chosen to approximate the major hysteresis loop.

The minor loop and the initial magnetization curve are bad.
Rotating field: \( h = H_m(t)(\cos t, \sin t) \). Same \( M_{an} \) and \( k \). The amplitude \( H_m(t) \) first grows, then remains constant.

The model solution \( \mathbf{m} = (m_x, m_y) \) (solid line) and the one based on the explicit approximation (dashed line) are close in this case.
The simplified model: Examples

Left: elliptic field
\[ h = H_m(t)(3 \cos t, \sin t); \]

Right: anisotropic case
\[ k = \text{diag}(k_0, 0.5k_0). \]

Explicit approximation (dashed line) vs model solution (solid line).
Modification 1: The composite Bergqvist model

One dry friction coefficient is replaced by a distribution of the pinning strength values.

In practice, a mixture of $N$ types of “pseudparticles” is assumed,

\[ m = \sum_{l=1}^{N} w^l m^l, \]

where the $l$-th particle type has the magnetization $m^l$ and volume fraction $w^l \geq 0$; $\sum_{l=1}^{N} w^l = 1$.

Here $m^l = M_{an}(h_r^l) \frac{h_r^l}{h_r^l}$ obeys the dry friction law and $h_r^l$ solves

\[ \min_{u \in K^l(t)} \{ S(u) - \hat{m}^l \cdot u \}, \]

where $K^l(t) := \{ u \in \mathbb{R}^3 : |\{ k^l \}^{-1}(h(t) - u)| \leq 1 \}$.

To account for the partial reversibility of material response set, say, $k^1 = 0$ and $K^1(t) := \{ h(t) \}$. 

Vector magnetic hysteresis model

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Modification 1: The composite Bergqvist model

Example: \( N = 20, \ w^l \equiv 1/N, \ k^l = 140 \frac{l-1}{N-1} \) A/m.

Improvement of the initial magnetization curve and minor hysteresis loops.
The idea: magnetic domains do not evolve independently driven by the magnetic field $h(t)$ but interact. The “driving force” is

$$h_{\text{eff}} = h + \alpha m,$$

where $\alpha$ is a material parameter. Such approach was used also for other models (J-A, Preisach, etc.); an explanation: Della Torre, 99.
Modification 2: The interaction term

With \( h_{\text{eff}} = h + \alpha m \) we find \( h_l^l \) for \( l = 1, \ldots, N \) solving

\[
\min_{u \in K^l(t,m)} \{ S(u) - \hat{m}^l \cdot u \}
\]

with the sets \( K^l(t) \) replaced by

\[
K^l(t, m) := \{ u \in \mathbb{R}^3 : |(k^l)^{-1}(u - h(t) - \alpha m)| \leq 1 \}
\]

if \( k^l \neq 0 \) and \( K^l(t, m) := \{ h(t) + \alpha m \} \) otherwise. Since

\[
m = \sum_{l=1}^{N} w^l m^l = \sum_{l=1}^{N} w^l M_{an}(h_r^l) \frac{h_r^l}{h_r^l}.
\]

the constraints in the opt. problems depend on the unknown solution: the whole problem becomes quasi-variational.
Identification of parameters

Parameters of the model:

- anhysteretic function $M_{an}$, a spline;
- weights $w^l$: assuming $N = 41$ and $k^l = 20 \times (l - 1) \text{ A/m}$ we seek $w^l$ s.t. $w^l \geq 0$, $\sum_{l=1}^{N} w^l = 1$;
- the material parameter $\alpha$.

A non-oriented steel: the experimental data and fitting results
Identification of parameters

The anhysteretic function $M_{an}$ (left) and weights $w^l$ (right). The reversible part ($k^1 = 0$) is strong; only 17 of 41 possible weights are nonzero;

$$\alpha = 8.8 \cdot 10^{-4}$$
We assumed the material is isotropic and included the hysteresis model with the identified parameters as a constitutive relation with memory,

\[ m = M[h_{\text{eff}}], \]

into the Maxwell equations. The local operator \( M \) keeps track of the internal variables \( h_r \) and \( m_l \) at each point of a ferromagnet.

A Newton-like iterations with the numerically approximated derivatives were used to treat this nonlinearity.

Another constitutive relation is the Ohm law, \( e = \rho j \).

The magnetization and eddy current losses, respectively, are:

\[ P_m = \int_0^t \int_\Omega \sum_{l=1}^N |r_l m_l'|, \quad P_j = \int_0^t \int_\Omega \rho |j|^2. \]
We solved a 2D eddy current and magnetization problem using an A-V formulation. Hence, the computation was confined to the domain of fm or conducting material.

The formulation is similar to that proposed in [d’Aquino et al. 2013] but employs the Bergqvist model for magnetization.

The geometrical configuration: hollow cylinder in a perpendicular field.

The fe mesh: 6424 triangles; 2352 triangles in the fm domain.
Simulation results

The external field first grows as $h_e(t) = \{10^3 t, 0\}$ A/m for 100 s, then rotates $90^\circ$ counter clockwise in the next 100 s.

Magnetic induction, $b$. 
Simulation results

Magnetization, $|m|$. Left: $t = 100$ s; right: $t = 200$ s.
Simulation results

Eddy current density, $j$. Left: $t = 100$ s; right: $t = 200$ s.

**Losses** (per unit of length):
- magnetization - 2.3 J/m, eddy current - 0.7 J/m.
Conclusions

The considered model

- is based on consistent energy arguments and a clear albeit simplified dry friction assumption;
- is vectorial, has a variational formulation, and can be employed in a f.e.m simulation;
- has sufficient degrees of freedom to be fitted to hysteretic behaviour of different materials;
- predicts the dissipation loss at any moment in time.

The typically employed simplifying approximation can be inaccurate and should be avoided.

Further comparison to experiments would be desirable.
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Thank you!